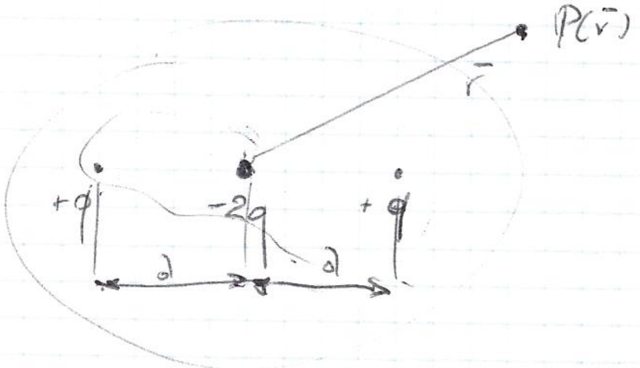
ES #12 Espansione multipolare di V; elettrostatica nei dielettrici

9/XII/2020



$$\sum_i q_i = \phi$$

$$\bar{p} = \sum q_i \bar{r}_i = \phi$$

$$V_0(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\,d\tau' \qquad f(\bar{r}') = \frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$\bar{r}' \to (0,0,0)$$

$$\rho(\bar{r}') \quad f(\bar{r}') \simeq f(\bar{r}'=\phi) +$$

$$+\left.\frac{\partial f}{\partial x'}\right|_{\bar{r}'=\phi}(x'-\phi) + \left.\frac{\partial f}{\partial y'}\right|_{\bar{r}'=\phi} y' + \left.\frac{\partial f}{\partial z'}\right|_{\bar{r}'=\phi} z' +$$

$$\sum_{\substack{i,j\\ i\neq j}} \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j + \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial x_i^2} x_i^2 + \dots$$

$$x_i = x, y, z$$

$$\left.\frac{\partial f}{\partial x'}\right|_{\bar{r}'=\phi} = \frac{x}{r^3} \qquad \text{simile per } y', z'$$

$$\left.\frac{\partial^2 f}{\partial x'^2}\right|_{\phi} = \frac{3x^2}{r^5} - \frac{1}{r^3} \quad ; \quad \left.\frac{\partial^2 f}{\partial x' \partial y'}\right|_{\phi} = \frac{3xy}{r^5}$$

(2)

$$f(\bar{r}') = f(x',y',z') = \frac{1}{r} +$$

$$+ \frac{xx'}{r^3} + \frac{yy'}{r^3} + \frac{zz'}{r^3} + \qquad \frac{\bar{r}\cdot\bar{r}'}{r^3} = \frac{\bar{r}'\cdot\bar{e}_r}{r^2}$$

$$+ \frac{1}{2}\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)x'^2 + \frac{1}{2}\left(\frac{3y^2}{r^5} - \frac{1}{r^3}\right)y'^2 + \frac{1}{2}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)z'^2 +$$

$$+ \frac{3xx'yy'}{r^5} + \frac{3xx'zz'}{r^5} + \frac{3yy'zz'}{r^5} \qquad -\frac{1}{2}\frac{r'^2}{r^3}$$

$$\frac{3}{2}\frac{(\bar{r}\cdot\bar{r}')^2}{r^5}$$

$$= \frac{1}{r} + \frac{\bar{r}\cdot\bar{r}'}{r^3} + \frac{3}{2}\frac{(\bar{r}\cdot\bar{r}')^2}{r^5} - \frac{1}{2}\frac{r'^2}{r^3}$$

$$V_0(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\rho(\bar{r}')\frac{1}{r}\,d\tau' + \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\rho(\bar{r}')\frac{\bar{r}\cdot\bar{r}'}{r^3}\,d\tau' +$$

$$+ \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\rho(\bar{r}')\left[\frac{3}{2}\frac{(\bar{r}\cdot\bar{r}')^2}{r^5} - \frac{1}{2}\frac{r'^2}{r^3}\right]d\tau'$$

termine di monopolo $\quad q \doteq \int_{\tau'}\rho(\bar{r}')\,d\tau' \quad \rightarrow \quad \sum_i q_i$

t. di dipolo $\quad \bar{p} = \int_{\tau'}\rho(\bar{r}')\bar{r}'\,d\tau' \quad \rightarrow \quad \sum_i q_i \bar{r}_i$

t. di quadrupolo $\quad Q = \int_{\tau'}\rho(\bar{r}')\left[\frac{3}{2}\frac{(\bar{r}\cdot\bar{r}')^2}{r^2} - \frac{r'^2}{2}\right]d\tau'$

$\quad (\bar{e}_r\cdot\bar{r}')^2$

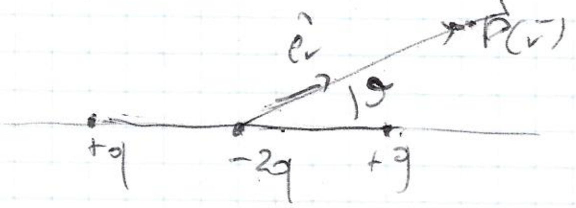
$$V_0(\bar{r}) \simeq \frac{1}{4\pi\varepsilon}\frac{q}{r} + \frac{1}{4\pi\varepsilon}\frac{\bar{p}\cdot\bar{r}}{r^3} + \frac{1}{4\pi\varepsilon}\frac{Q}{r^3} =$$

(3)

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{\bar{p}\cdot\hat{e}_r}{r^2} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}$$

pt fonni

$$Q = \sum_i q_i \left[\frac{3}{2}(\bar{r}_i\cdot\hat{e}_r)^2 - \frac{r_i^2}{2}\right]$$



$q = \phi$

$\bar{p} = \phi$

$$Q = +q\left[\frac{3}{2}(a\cos\vartheta)^2 - \frac{a^2}{2}\right] +$$

$$-2q\left(\frac{3}{2}\phi^2 - \frac{1}{2}\phi\right) +$$

$$+q\left[\frac{3}{2}(a\cos(\pi-\vartheta))^2 - \frac{1}{2}a^2\right] = 3qa^2\cos^2\vartheta - qa^2 =$$

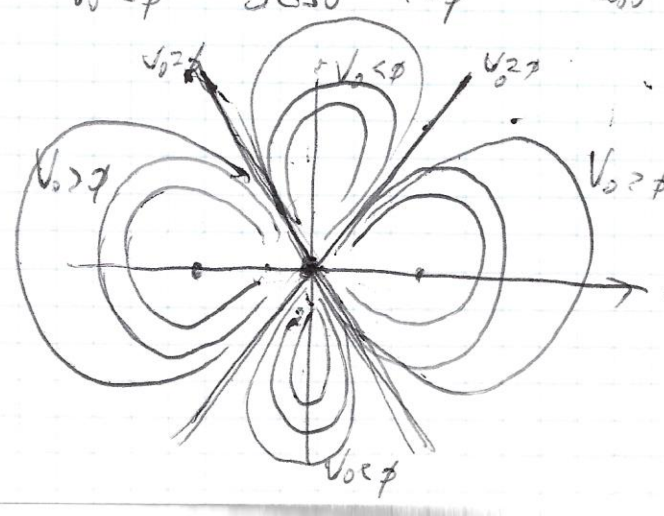
$(-a\cos\vartheta)^2$

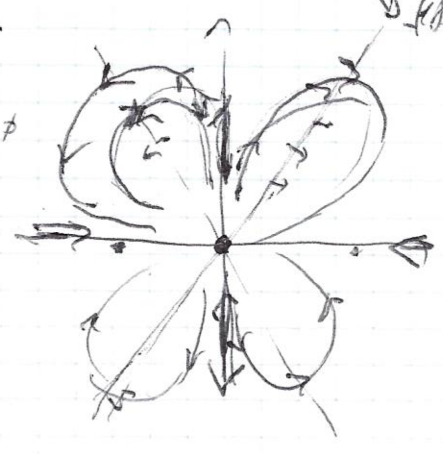
$$= \underline{qa^2(3\cos^2\vartheta - 1)}$$

$$V_0(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qa^2(3\cos^2\vartheta - 1)}{r^3}$$

$V_0 = \phi \quad 3\cos^2\vartheta - 1 = \phi \quad \cos\vartheta = \pm\sqrt{3}/3 \quad \vartheta = \pm 54.74^\circ$

$\pm(180^\circ - 54.74^\circ) = \pm 125.26^\circ$





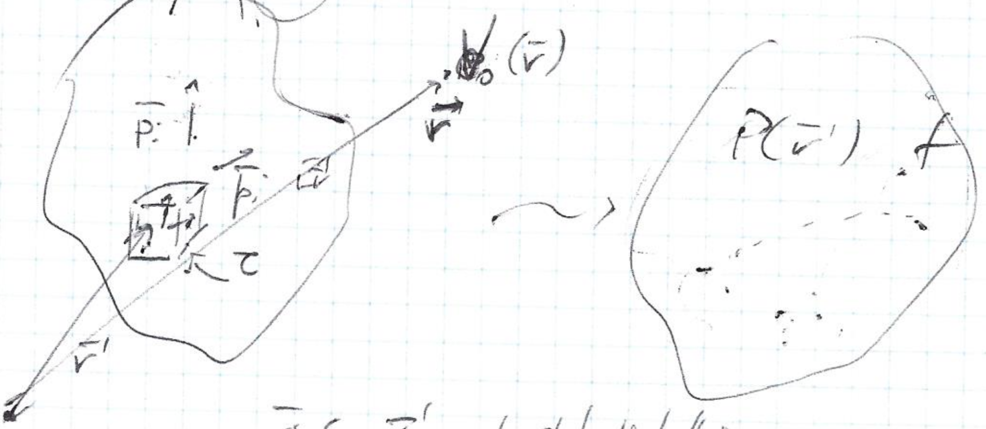
Dielettrici

Polarizzazione

$\overline{P}$ vettore (campo vettoriale) macroscopico
densità di polarizzazione

micro- → macro- → equivalenza $V_0, \overline{E_0}$
↑
distrib. equiv.
di carica NEL VUOTO

$$\overline{P}(\overline{r}') = \lim_{\tau \to 0} \frac{\sum \overline{p_i}}{\tau} \quad (= N\overline{p})$$



$\overline{r} \in \tau_0$ vol. del dielettrico

$$d\tau \rightsquigarrow d\overline{p}(\overline{r}') = \overline{P}\, d\tau' \qquad \overline{P} = \frac{d\overline{p}}{d\tau}$$

$$V_0(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int \underbrace{\frac{\overline{P}(\overline{r}') \cdot (\overline{r}-\overline{r}')}{|\overline{r}-\overline{r}'|^3}}_{dV_0(\overline{r})} d\tau' =$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{S=\partial\tau} \frac{\overline{P} \cdot \hat{e}_n}{|\overline{r}-\overline{r}'|} dS + \frac{1}{4\pi\varepsilon_0} \int_{\tau} \frac{-\overline{\nabla}' \cdot \overline{P}}{|\overline{r}-\overline{r}'|} d\tau'$$

$$\begin{cases} \sigma_p = \overline{P} \cdot \hat{e}_n \\ \rho_p = -\overline{\nabla} \cdot \overline{P} \end{cases}$$

Per spazio vuoto con σ_p, ρ_p ⑤

$$V_0(\bar r) = \frac{1}{4\pi\varepsilon_0}\int_{S=\partial\tau} \frac{\sigma_p(\bar r')}{|\bar r-\bar r'|}\,d\tau' + \frac{1}{4\pi\varepsilon_0}\int_{\tau} \frac{\rho_p(\bar r')}{|\bar r-\bar r'|}\,d\tau'$$

EQUIVALENZA
$$\begin{cases} \sigma_p = \bar P\cdot\hat e_n \\ \rho_p = -\bar\nabla\cdot\bar P \end{cases}$$

$\bar P = f(\bar E)$ rel. costitutiva

$\bar P = \varepsilon_0 \chi \bar E$ $\quad \chi$ suscettività dielettrica

$\chi = f(\bar E, \text{materiale})$

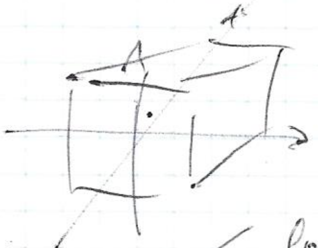
χ tensore (matrice)

dielettrici lineari, omogenei, isotropi

linearità $\bar P - \bar E$ lineare $\bar P = \alpha \bar E$ $\quad \chi = \text{cost}$

omogeneità χ indip. da $\bar r$; se $\chi = \chi(\bar r)$ disomogeneo

isotropia risposta del mat. indip. direzione



$\bar P \not\parallel \bar E$ anisotropo

$$\bar\nabla\cdot\bar E = \frac{\rho+\rho_p}{\varepsilon_0} \quad \rho_p = -\bar\nabla\cdot\bar P \quad \leadsto \quad \bar\nabla\cdot(\underbrace{\varepsilon_0\bar E+\bar P}_{\bar D}) = \rho$$

$$\bar D = \varepsilon_0\bar E + \bar P$$

$$\bar\nabla\cdot\bar D = \rho \quad \leadsto \quad \Phi(\bar D) = \oint_S \bar D\cdot d\bar S = Q$$

Diel. lineare

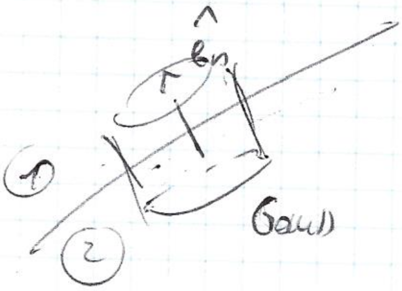
$$\bar{P} = \varepsilon_0 \chi \bar{E} \qquad \bar{D} = \varepsilon_0(\chi+1)\bar{E} = \varepsilon_0 \varepsilon_r \bar{E} =$$

$$\varepsilon_r = K = \chi + 1 \qquad = \varepsilon_0 K \bar{E}$$

cost. dielettrica relativa

$$\varepsilon = \varepsilon_0 K = \varepsilon_0 \varepsilon_r$$

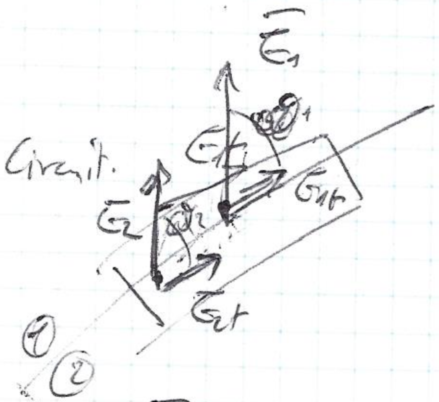
Cond. di raccordo



$$\bar{\nabla} \cdot \bar{D} = \rho$$

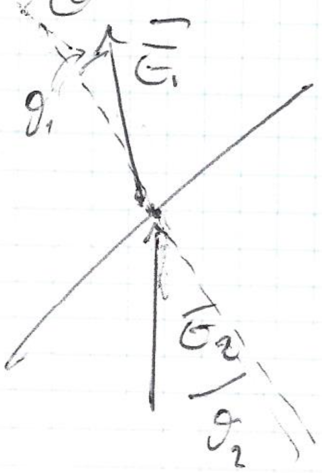
$$\oint(\bar{D}) = \int_S \bar{D} \cdot d\bar{S} = Q_{INT} = 0$$

$$D_{1n} = D_{2n}$$



$$\oint \bar{E} \cdot d\bar{l} = 0 \qquad (\bar{\nabla} \times \bar{E} = 0)$$

$$\bar{E}_{1t} = \bar{E}_{2t}$$

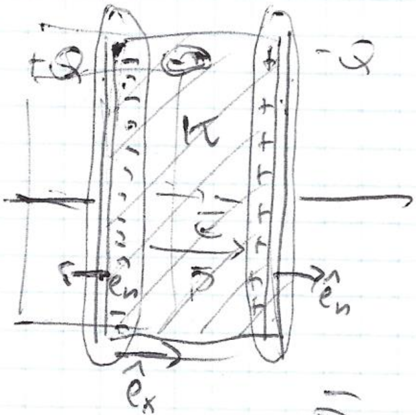


$$\frac{E_{1t}}{D_{1n}} = \frac{E_{2t}}{D_{2n}}$$

$$\frac{E_{1t}}{\varepsilon_1 E_{1n}} = \frac{E_{2t}}{\varepsilon_2 E_{2n}} \qquad \frac{tg\,\vartheta_1}{tg\,\vartheta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{K_1}{K_2}$$

rifrazione di $\bar{E}$

Cond. piano κ lin., omo-, iso-



$$\vec{E} \to \vec{D}$$

Gauss

$$\Phi(\vec{D}) = \int_S \vec{D}\cdot d\vec{S} = Q$$

$$\vec{D} = \sigma \hat{e}_x = \frac{Q}{S}\hat{e}_x$$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \kappa \vec{E}$$

$$\Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{\varepsilon S}\hat{e}_x = \frac{\sigma}{\varepsilon}\hat{e}_x \left(= \frac{\vec{E}_0}{\kappa}\right)$$

$$\Delta V = \int_0^d E_x\, dx = \frac{Qd}{\varepsilon_0 \kappa S} \left(= \frac{\Delta V_0}{\kappa}\right)$$

$$C = Q/\Delta V = \varepsilon_0 \kappa S/d = \kappa C_0$$

Q pol.?

$$\rho_p = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\chi \varepsilon_0 \vec{E}) = -\chi\varepsilon_0 \vec{\nabla}\cdot\vec{E} = 0$$

sx

$$\sigma_p^{sx} = \vec{P}\cdot\hat{e}_n = \varepsilon_0 \chi \underbrace{\vec{E}}_{\parallel \hat{e}_x}\cdot(-\hat{e}_x) = -(\kappa-1)\varepsilon_0 \frac{\sigma}{\varepsilon_0 \kappa} = -\frac{\kappa-1}{\kappa}\sigma$$

$$\sigma_p(S) = \vec{P}(S)\cdot\hat{e}_n(S)$$

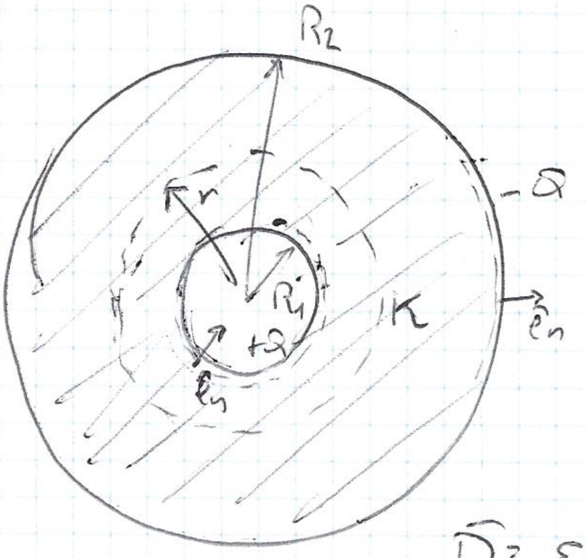
$$\cancel{\sigma_p(\vec{r}) = \vec{P}\cdot\hat{e}_n}$$

$$\sigma_p^{dx} = \vec{P}\cdot\hat{e}_n = \vec{P}\cdot\hat{e}_x = \frac{\kappa-1}{\kappa}\sigma$$

$$Q_p = \sigma_p^{sx}\cdot S + \sigma_p^{dx}\cdot S \left(+\int \rho_p\, d\tau\right) = 0$$

globalmente neutro

$$\text{equiv. } Q' = Q_{free}^{sx} + Q_p^{sx} = S\left(\sigma - \frac{\kappa-1}{\kappa}\sigma\right) = \frac{Q}{\kappa}$$



$R_1 < r < R_2$

$$\oint \vec{D} \cdot d\vec{S} = Q$$

$$D_r(r) \cdot 4\pi r^2 = Q \Rightarrow D_r(r) = \frac{Q}{4\pi r^2}$$

$$\vec{D} = \varepsilon \vec{E}$$

$$E_r(r) = \frac{1}{4\pi\varepsilon_0 K} \frac{Q}{r^2}$$

$$\Delta V = V_1 - V_2 = \int_{R_1}^{R_2} E_r dr = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon} \frac{R_2 - R_1}{R_1 R_2} \left(= \frac{\Delta V_0}{K}\right)$$

$$C = \frac{Q}{\Delta V} = 4\pi\varepsilon \frac{R_1 R_2}{R_1 + R_2} = K C_0$$

$$\rho_p = -\vec{\nabla}\cdot\vec{P} \propto \vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E_r\right) = \emptyset$$

$$\rho_p = \emptyset$$

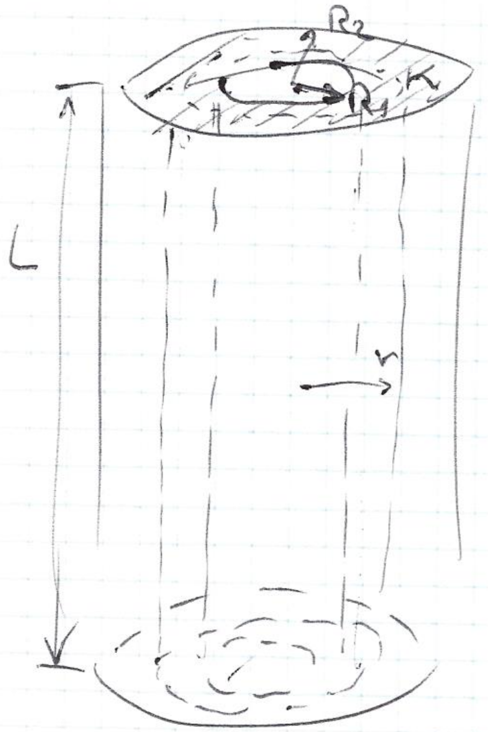
$$\sigma_p(R_1) = \vec{P}(R_1)\cdot\hat{e}_n(R_1) = \vec{P}(R_1)\cdot(-\hat{e}_r) = -(K-1)\varepsilon_0 E_r(R_1) =$$

$$= -(K-1)\varepsilon_0 \frac{1}{4\pi\varepsilon_0 K}\frac{Q}{R_1^2} = -\frac{(K-1)}{K}\sigma$$

$$\sigma_p(R_2) = \vec{P}(R_2)\cdot\hat{e}_r = (K-1)\varepsilon_0 \frac{1}{4\pi\varepsilon_0 K}\frac{Q}{R_2^2}\cdot\left(\frac{R_1^2}{R_1^2}\right) =$$

$$= \frac{K-1}{K}\frac{R_1^2}{R_2^2}\sigma$$

$$\left.\begin{aligned} Q_{p1} &= \sigma_p^1 \cdot 4\pi R_1^2 = -\frac{K-1}{K} Q \\ Q_{p2} &= \sigma_p^2 \cdot 4\pi R_2^2 = +\frac{K-1}{K} Q \end{aligned}\right\} Q_p = \emptyset$$



$$\int_S \bar{D}\cdot d\bar{S} = Q$$

$$D_r(r)\, 2\pi r L = Q \quad (= 2\pi R_1 L \sigma)$$

$$\bar{D}(r) = \frac{1}{2\pi L}\frac{Q}{r}\hat{e}_r$$

$$\bar{D} = \varepsilon\bar{E} \Rightarrow \bar{E}(r) = \frac{1}{2\pi\varepsilon L}\frac{Q}{r}\hat{e}_r$$

$$\Delta V = V_1 - V_2 = \int_{R_1}^{R_2} E_r(r)\,dr = \frac{Q}{2\pi \underbrace{\varepsilon}_{\varepsilon_0 K} L}\log(R_2/R_1) \quad \left(= \frac{\Delta V_0}{K}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon L}{\log(R_2/R_1)} = K C_0$$

ε_r

$$\rho_p = -\bar{\nabla}\cdot\bar{P} = \phi \qquad \bar{\nabla}\cdot\bar{E} \sim \frac{1}{r}\frac{d}{dr}(rE_r) = \phi$$

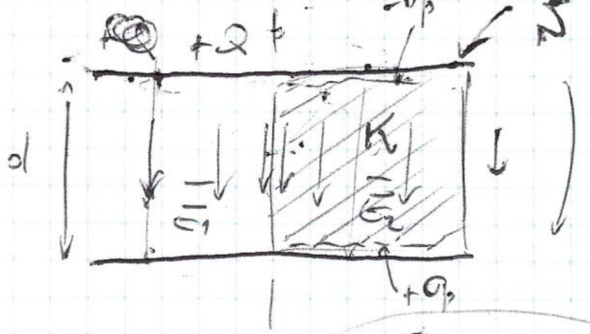
$\frac{1}{r}$

$$\sigma_p^1 = \sigma_p(R_1) = \bar{P}(R_1)\cdot\underbrace{\hat{e}_n(R_1)}_{-\hat{e}_r} = -(K-1)\varepsilon_0\frac{1}{2\pi\varepsilon_0 K L}\frac{Q}{R_1} = -\frac{K-1}{K}\sigma$$

$$\sigma_p^2 = \sigma_p(R_2) = \bar{P}(R_2)\cdot\underbrace{\hat{e}_n(R_2)}_{\hat{e}_r} = (K-1)\varepsilon_0\frac{1}{2\pi\varepsilon_0 K L}\frac{Q}{R_2} = \frac{K-1}{K}\frac{R_1}{R_2}\sigma$$

$$Q_p^1 = \sigma_p(R_1)\cdot 2\pi R_1 L = -\frac{K-1}{K}Q$$

$$Q_p^2 = \sigma_p(R_2)\cdot 2\pi R_2 L = \frac{K-1}{K}Q$$



ΔV

E unif., $\perp$ armature

soddisfa le b.c., i.b.c.

$$\Delta V = E_1 d = E_2 d \qquad E_1 = E_2 = E$$

$$\begin{cases} E = E_1 = \sigma_1/\varepsilon_0 & \sigma_1 \text{ q libera} \\ E = E_2 = (\sigma_2 - \sigma_p)/\varepsilon_0 & \sigma_2 \text{ q libera} \end{cases}$$

$$\sigma_1 = \varepsilon_0 E$$

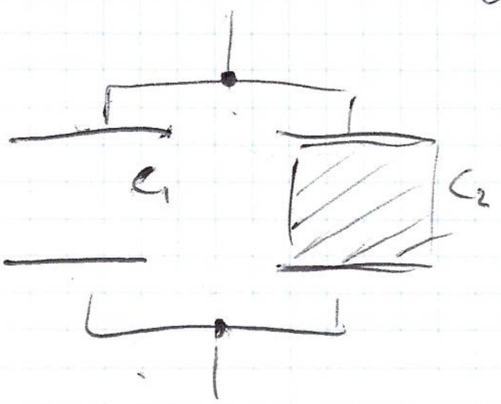
$$\sigma_2 - \sigma_p = \varepsilon_0 E \qquad \text{ma} \qquad \sigma_p = \vec{P}\cdot\hat{e}_n = P = \varepsilon_0 (K-1) E$$

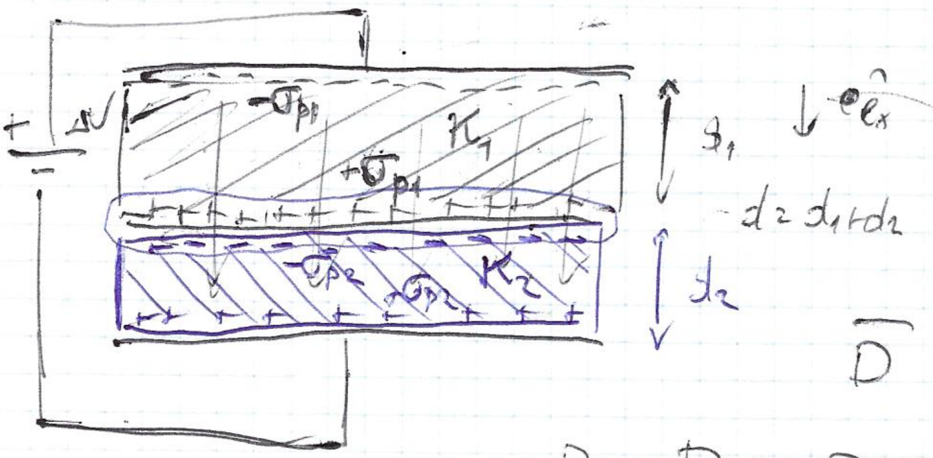
$$\underline{\sigma_2} = \varepsilon_0 E + \sigma_p = \varepsilon_0 E + \varepsilon_0 (K-1) E = \underline{\varepsilon_0 K E}$$

$$\begin{cases} \sigma_1 = \varepsilon_0 \Delta V/d & Q_1 = \sigma_1 \cdot S/2 \\ \sigma_2 = \varepsilon_0 K \Delta V/d = K\sigma_1 & Q_2 = \sigma_2 S/2 = K\sigma_1 S/2 \end{cases}$$

$$Q = Q_1 + Q_2 = \varepsilon_0 (K+1) \Delta V \cdot S/2d$$

$$C = \frac{Q}{\Delta V} = \frac{\varepsilon_0 (K+1) S}{2d} = \frac{\varepsilon_0 S}{2d} + \frac{\varepsilon_0 K S}{2d} = \underset{\text{vuoto}}{C_1} + \underset{\text{diel.}}{C_2}$$





$\bar{D}$ ⊥ armature nel cond.

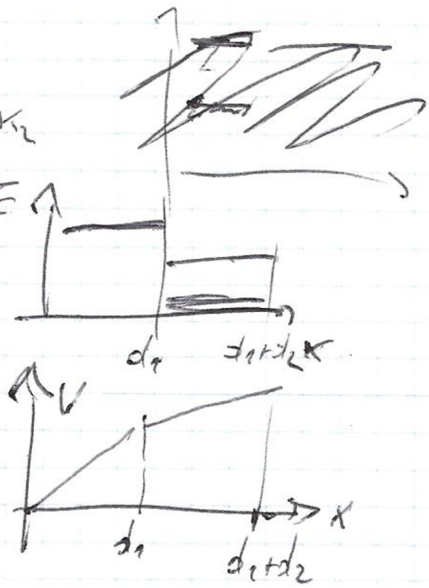
$D = D_{1n} = D_{2n}$ uniforme nel cond.

$D = \frac{Q}{S} = \sigma$ $\qquad \bar{D}_i = \varepsilon_i \bar{E}_i \qquad i = 1, 2$

$$E_1 = \frac{D}{\varepsilon_1} = \frac{Q}{\varepsilon_0 K_1 S} = \frac{\sigma}{\varepsilon_0 K_1}$$

$K_1 < K_2$

$$E_2 = D/\varepsilon_2 = \frac{\sigma}{\varepsilon_0 K_2}$$



$$\Delta V = \int \bar{E} \cdot d\bar{l} = E_1 d_1 + E_2 d_2 = \frac{Q}{\varepsilon_0 S}\left(\frac{d_1}{K_1} + \frac{d_2}{K_2}\right)$$

$$C_{tot} = \frac{Q}{\Delta V} = \frac{\varepsilon_0 S}{\left(\frac{d_1}{K_1} + \frac{d_2}{K_2}\right)} = \frac{S \varepsilon_0 K_1 K_2}{d_1 K_2 + d_2 K_1}$$

$$\frac{1}{C_{tot}} = \frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{(serie)}$$