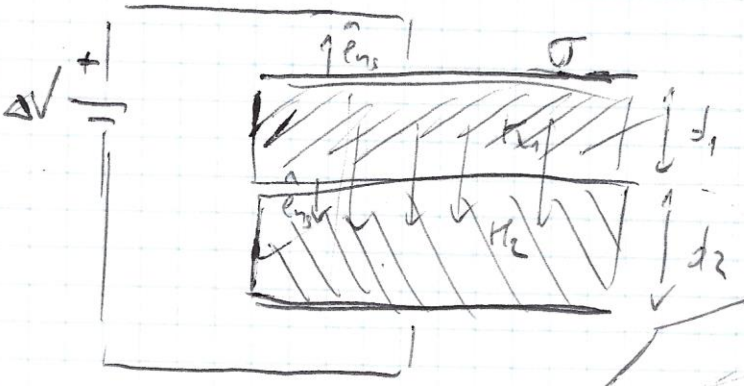
ES. #13 Dielettrici; condensatori con dielettrico; energia e forze 11/XII/2020 (1)



$$\hat{e}_n \quad D_{(n)} = \frac{Q}{S} = \sigma$$

$$E_1 = \frac{D}{\varepsilon_1} = \frac{\sigma}{\varepsilon_0 K_1}$$

$$E_2 = \frac{D}{\varepsilon_2} = \frac{\sigma}{\varepsilon_0 K_2}$$

$$\Delta V = \int \bar{E} \cdot d\bar{l} = E_1 d_1 + E_2 d_2$$

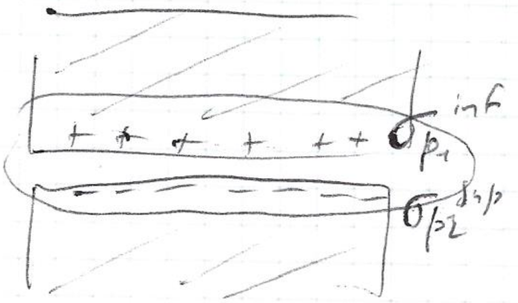
$$C = \frac{Q}{\Delta V} = \varepsilon_0 S \Big/ \left(\frac{d_1}{K_1} + \frac{d_2}{K_2} \right) = \frac{S \varepsilon_0 K_1 K_2}{d_1 K_2 + d_2 K_1}$$

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d_1}{\varepsilon_0 K_1 S} + \frac{d_2}{\varepsilon_0 K_2 S}$$

$$\bar{\nabla} \cdot \bar{P} = \phi \qquad \rho_{p1},\ \rho_{p2} = \phi \qquad \bar{P} = \varepsilon_0 \chi \bar{E}$$

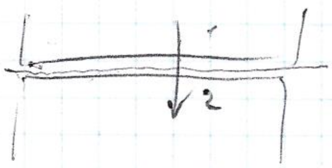
$$\sigma_{p1}^{sup,inf} = \bar{P}_1 \cdot \hat{e}_{ns1}^{sup,inf} = \mp \frac{K_1 - 1}{K_1} \sigma$$

$$\sigma_{p2}^{sup,inf} = \bar{P}_2 \cdot \hat{e}_{ns2}^{sup,inf} = \mp \frac{K_2 - 1}{K_2} \sigma$$



$$\sigma_{p\,netta} = \sigma_{p1}^{inf} + \sigma_{p2}^{sup}$$

i.b.c. interface boundary condition



$$\Delta E_n = E_2 - E_1 = \sigma/\varepsilon_0$$

$$E_2 - E_1 = \frac{\sigma}{\varepsilon_0}\left(\frac{1}{K_2} - \frac{1}{K_1}\right) = \frac{\sigma(K_1-K_2)}{\varepsilon_0 K_1 K_2}$$

$$\sigma/\varepsilon_0 = \frac{1}{\varepsilon_0}\left(\sigma_{p1}^{inf} + \sigma_{p2}^{sup}\right) = \frac{1}{\varepsilon_0}\sigma\left(\frac{K_1-1}{K_1} - \frac{K_2-1}{K_2}\right) = \frac{\sigma}{\varepsilon_0}\left(\frac{K_1-K_2}{K_1K_2}\right)$$

Rigidità dielettrica: E_{max} sostenibile prima di condurre

1. carta $K_1 = 2$; $d_1 = 2.5$ mm $E_{R1} = 6\cdot 10^6$ V/m
2. porcellana $K_2 = 4$; $d_2 = 1.5$ mm $E_{R2} = 12\cdot 10^6$ V/m

$E_1 \le E_{R1}$; $E_2 \le E_{R2}$

$$E_1 = \frac{Q}{\varepsilon_1 S} = \frac{C\Delta V}{\varepsilon_1 S} = \frac{K_2}{d_1K_2 + d_2K_1}\Delta V < E_{R1}$$

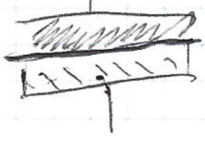
$$\Delta V_1 < 19.5 \text{ kV}$$

$$E_2 = \frac{Q}{\varepsilon_2 S} = \frac{C\Delta V}{\varepsilon_2 S} = \frac{K_1}{d_1K_2 + d_2K_1}\Delta V < E_{R2}$$

$$\Delta V_2 < 78 \text{ kV}$$

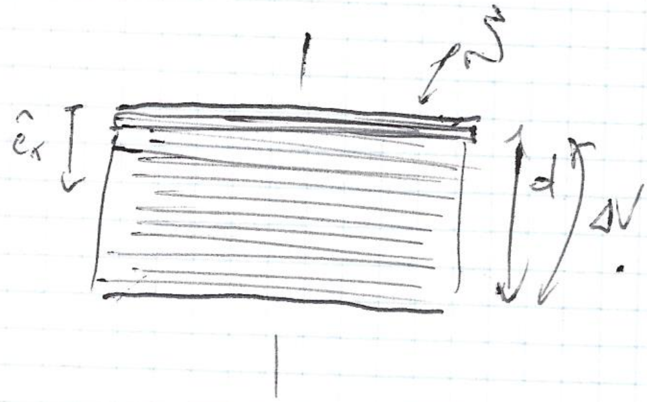
Cond. di sicurezza $\Delta V < \min(1,2) = 19.5$ kV

Bruciato K_1



$$E = \frac{\Delta V}{d_2} < E_{R2}$$

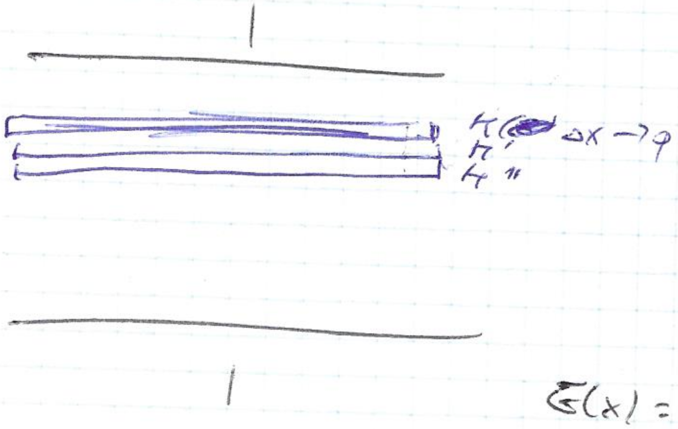
$$\Delta V_{max} < E_{R2}\, d_2 = 18 \text{ kV}$$



$$K(x) = 1/(1-\alpha x^2) \quad (\alpha \text{ cost.})$$

$$\Delta V = \int_0^d E(x)\,dx$$

D_n uniforme



$$D = \sigma = Q/S$$

$$\bar{D}_{(x)} = \varepsilon_0 K_{(x)} \bar{E}(x)$$

$$E(x) = \frac{D}{\varepsilon_0 K(x)} = \frac{\sigma}{\varepsilon_0}(1-\alpha x^2)$$

$$\Delta V = \frac{\sigma}{\varepsilon_0}\int_0^d (1-\alpha x^2)\,dx = \frac{\sigma}{\varepsilon_0}\left[x - \frac{\alpha x^3}{3}\right]_0^d = \frac{\sigma}{\varepsilon_0}\left(d - \alpha d^3/3\right)$$

$$C = Q/\Delta V = \frac{\varepsilon_0 S}{d-\alpha d^3/3}$$

$$Q = \sigma S$$

$$C_{\text{ennesimo}} = \frac{\varepsilon_0 K(x) S}{dx} \quad ; \quad \frac{1}{C_{tot}} = \sum \frac{1}{C_{\text{ennesimo}}}$$

$$\frac{1}{C_{\text{ennesimo}}} = \frac{1}{\varepsilon_0 S}\frac{dx}{K(x)} = \frac{1}{\varepsilon_0 S}(1-\alpha x^2)\,dx$$

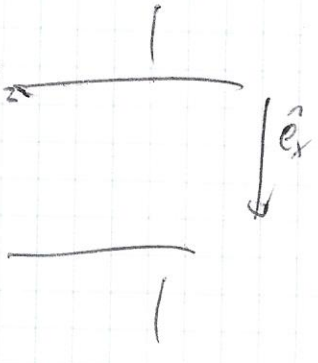
$$\frac{1}{C_{tot}} = \frac{1}{\varepsilon_0 S}\int_0^d (1-\alpha x^2)\,dx = \frac{1}{\varepsilon_0 S}\left(d - \alpha d^3/3\right)$$

4

$$\rho_p(x) = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\varepsilon_0(\kappa(x)-1)\vec{E}(x)\right) = \quad \vec{E} = \vec{D}/\varepsilon_0\kappa = \frac{\sigma}{\varepsilon_0\kappa}\hat{e}_x$$

$$= -\vec{\nabla}\cdot\left[\frac{\kappa-1}{\kappa}\vec{D}\right] = -\frac{d}{dx}\left[\frac{\kappa(x)-1}{\kappa(x)}\sigma\right] =$$

$$= -\sigma\frac{d}{dx}(\alpha x^2) = -2\sigma\alpha x$$



$$\sigma_p(0) = \vec{P}(0)\cdot\underbrace{\hat{e}_n}_{-\hat{e}_x} = \varepsilon_0(\kappa(0)-1)\vec{E}(0)\cdot(-\hat{e}_x) =$$

$$= -\varepsilon_0\underbrace{(1-1)}\frac{\sigma}{\varepsilon_0}(1-\alpha\cancelto{0}{x}^2) = 0$$

$$\sigma_p(d) = \vec{P}(d)\cdot\hat{e}_n = \varepsilon_0(\kappa(d)-1)\vec{E}(d)\cdot\hat{e}_x =$$

$$= \varepsilon_0\left(\frac{1}{1-\alpha d^2}-1\right)\frac{\sigma}{\varepsilon_0}\cancel{(1-\alpha d^2)} = \sigma\alpha d^2$$

$$Q_{p\,vol} = \int \rho_p\, d\tau = \int_0^d -2\sigma\alpha x\underbrace{S\,dx}_{d\tau} = \left[-\sigma\alpha S x^2\right]_0^d$$

$$\Rightarrow Q_{p\,vol} = -\sigma\alpha S d^2$$

$$\sigma_p(0) = 0 \quad ; \quad \sigma_p(d) = \sigma\alpha d^2$$

$$(+) = 0$$

$$Q_{p\,sup} = \sigma\alpha d^2 S$$

Energia (con dielettrici)

5

$$U = \frac{1}{2}\int_\tau \rho V d\tau = \frac{1}{2}\int_\tau (\vec{\nabla}\cdot\vec{D}) V d\tau = \frac{1}{2}\int_\tau \vec{\nabla}\cdot(V\vec{D}) d\tau + \frac{1}{2}\int_\tau \vec{E}\cdot\vec{D} d\tau =$$

$$= \frac{1}{2}\int_{S=\partial\tau} V\vec{D}\cdot d\vec{S} + \frac{1}{2}\int_\tau \vec{E}\cdot\vec{D} d\tau$$

$\tau \to \infty$

distr. di carica limitata

$V\vec{D} \to \phi$

$\vec{r} \to \infty$

$$U = \frac{1}{2}\int_\tau \boxed{\vec{E}\cdot\vec{D}} d\tau$$

$V\vec{D}\tau \to \phi$

$r \to \infty$

$$u = \frac{1}{2}\vec{E}\cdot\vec{D}$$

$\vec{D} = \varepsilon\vec{E}$ $(\varepsilon_0 \vec{E})$

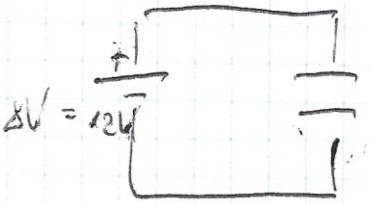
$$u = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\frac{D^2}{\varepsilon}$$

$\varepsilon > \varepsilon_0$

lavoro di polarizzazione

$$\frac{1}{2}\varepsilon_0 \vec{E}\cdot\vec{E}$$

A tensione costante



$C_0 = 100\,nF$ $\quad\Delta V = 12\,V$

$K = 4$

$C = K C_0 = 400\,nF$

$$\begin{cases} Q_i = C_0 \Delta V = 1.2\,\mu C \\ Q_f = C_f \Delta V = K C_0 \Delta K = 4.8\,\mu C \end{cases}$$

$\Delta Q = (K-1) C_0 \Delta V = 3.6\,\mu C$

$$U_{ci} = \frac{1}{2} C_0 (\Delta V)^2 = \frac{1}{2}\frac{Q_i^2}{C} = 7.2\,\mu J$$

$$U_{cf} = \frac{1}{2} C_f (\Delta V)^2 = K U_{ci} = 28.8\,\mu J$$

$$\Delta U_c = U_{cf} - U_{ci} = 21.6\,\mu J$$

Gen.: ΔQ

$$W = \underset{12V}{(\Delta V)} \cdot \Delta Q = (K-1) C_0 (\Delta V)^2 = 2(\Delta U_c)$$

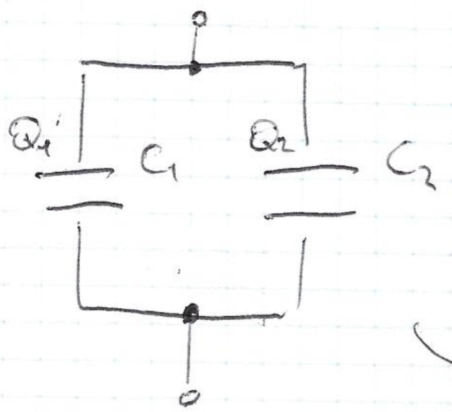
A carica costante

$Q_f = Q_i = C_0 \Delta V = 1.2\,\mu C$

$$U_{ci} = \frac{1}{2} C_0 (\Delta V)^2 = \frac{1}{2}\frac{Q_i^2}{C_0} = 7.2\,\mu J$$

$$U_{cf} = \frac{1}{2}\frac{Q_f^2}{C_f} = \frac{1}{2}\frac{Q_i^2}{K C_0} = 1.8\,\mu J$$

Carica costante



$$C_{eq} = C_1 + C_2$$

$$\Delta V_i = \frac{Q}{C_{eq}} = \frac{Q}{C_1+C_2}$$

$$Q_1 = C_1 \Delta V_i = \frac{C_1}{C_1+C_2} Q$$

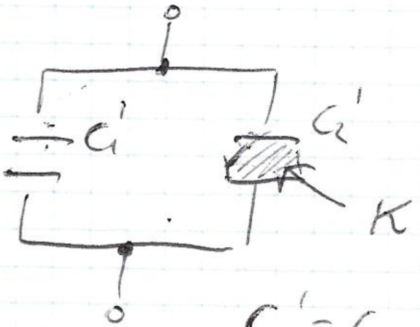
$$Q_2 = C_2 \Delta V_i = \frac{C_2}{C_1+C_2} Q$$

partitore capacitivo di carica

$$U_i = \frac{1}{2}\frac{Q^2}{C_{eq}} = \frac{1}{2}\frac{Q^2}{C_1+C_2}$$

$$C_1 \neq C_2$$

$$Q = Q_1 + Q_2$$



$$C_1' = C_1$$

$$C_2' = K C_2$$

$$C_{eq}' = C_1' + C_2' = C_1 + K C_2$$

$$\Delta V_f = \frac{Q}{C_{eq}'} = \frac{Q}{C_1 + K C_2} < \Delta V_i$$

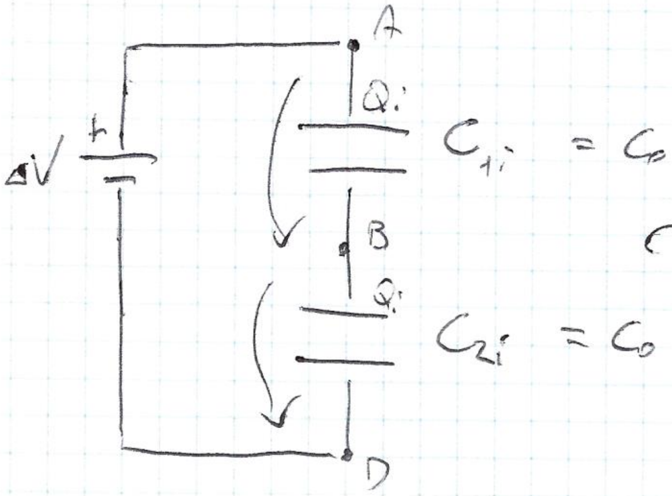
$$Q_1' = C_1 \Delta V_f = \frac{C_1}{C_1 + K C_2} Q$$

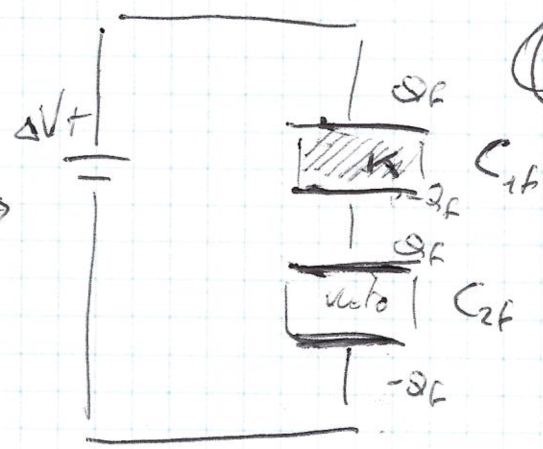
$$Q_2' = C_2' \Delta V_f = K C_2 \Delta V_f = \frac{K C_2}{C_1 + K C_2} Q$$

$$Q_1' + Q_2' = Q$$

$$U_f = \frac{1}{2}\frac{Q^2}{C_{eqf}} = \frac{1}{2}\frac{Q^2}{C_1 + K C_2} < U_i$$

9





$$\frac{1}{C_{eq_i}} = \frac{1}{C_0} + \frac{1}{C_0} = \frac{2}{C_0}$$

$$\Rightarrow C_{eq_i} = C_0/2$$

$$Q_i = C_{eq_i}\Delta V = \frac{C_0 \Delta V}{2}$$

$$(V_A - V_B)_i = \frac{Q_i}{C_{1i}} = \frac{C_{eq_i}\Delta V}{C_{1i}} = \frac{\Delta V}{2}$$

$$(V_B - V_D)_i = \frac{Q_i}{C_{2i}} = \frac{C_{eq_i}\Delta V}{C_{2i}} = \frac{\Delta V}{2}$$

$$U_i = \frac{1}{2} C_{eq_i}(\Delta V)^2 = \frac{1}{4} C_0 (\Delta V)^2$$

$$C_{1f} = K C_{1i} = K C_0$$

$$C_{2f} = C_0$$

$$\frac{1}{C_{eq_f}} = \frac{1}{C_{1f}} + \frac{1}{C_{2f}} = \frac{K+1}{K C_0}$$

$$C_{eq_f} = \frac{K}{K+1} C_0 > C_{eq_i} = \frac{C_0}{2}$$

$$Q_f = C_{eq_f}\Delta V$$

$$(V_A - V_B)_f = \frac{Q_f}{C_{1f}} = \frac{C_{eq_f}\Delta V}{C_{1f}} = \frac{\Delta V}{K+1} < \frac{\Delta V}{2}$$

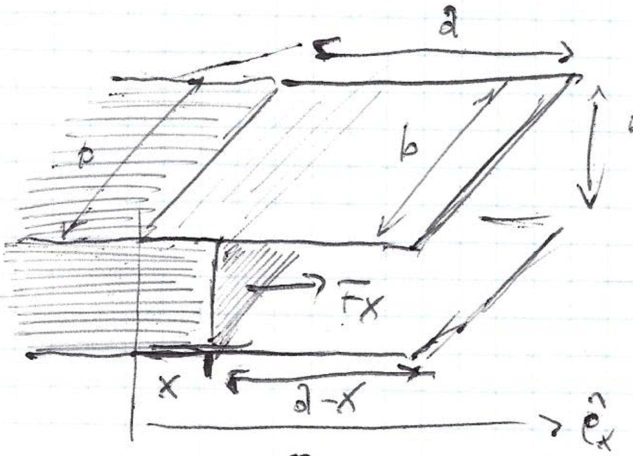
$$(V_B - V_D)_f = \frac{C_{eq_f}\Delta V}{C_{2f}} = \frac{K}{K+1}\Delta V > \frac{\Delta V}{2}$$

$$Q_1 = C_{1f}\,\Delta V/(K+1) =$$

$$Q_2 = C_{2f}\Delta V/(K+1) =$$

$$K C_0 \Delta V/(K+1)$$

$$U_f = \frac{1}{2} C_{eq_f}(\Delta V)^2 = \frac{1}{2}\,\frac{K}{K+1}\, C_0 (\Delta V)^2$$

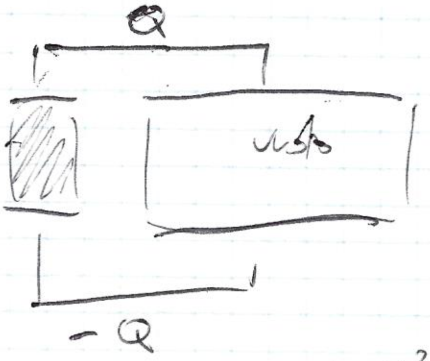


10

Q costante

Q sul cond., scollegato

poi → inserisco dielettrico (K)



$$C = \frac{\varepsilon_0 K b x}{d} + \frac{\varepsilon_0 b(a-x)}{d}$$

$$C = \frac{\varepsilon_0 b}{d}\left[a + x(K-1)\right]$$

$$U_c = \frac{1}{2}\frac{Q^2}{C(x)} = Q^2 d \Big/ 2\varepsilon_0 b\left[a+x(K-1)\right]$$

$$F_x = -\frac{\delta U_c}{\delta x} = -\left(\vec{\nabla} U\right)_x = -\frac{\partial U_c}{\partial x} = \frac{Q^2 d}{2\varepsilon_0 b}\frac{K-1}{\left[a+x(K-1)\right]^2}$$

$/\!/ \hat{e}_x \longrightarrow$

$$E = \Delta V/d = \frac{Q}{Cd} \Rightarrow E = \frac{Q}{\varepsilon_0 b\left(a+x(K-1)\right)}$$

$$F_x = \frac{1}{2}E^2(x)\,\varepsilon_0 b d (K-1)$$

$a = 50$ mm	$K = 3$	$C = 6.37$ pF
$b = 40$ mm	$Q = 1$ nC	$F_x = 1.74\ \mu$N
$d = 5$ mm		$E = 3.14 \cdot 10^4$ V/m
$x = \frac{1}{2}b = 20$ mm		$\Delta V = 156{,}87$ V

Processo a tensione costante V tra le armature

$$U_c = \frac{1}{2} C(x) V^2 = \frac{1}{2} V^2 \frac{\varepsilon_0 b}{d} \left[a + x(K-1)\right]$$

$$dx \qquad dU_c = \frac{1}{2} V^2 dC(x) = \frac{1}{2} V^2 \frac{\varepsilon_0 b}{d} (K-1)\, dx$$

tensione costante $\Rightarrow$ sistema $\left\{ \begin{array}{l} \text{Cond.} \\ + \\ \text{gen.} \end{array} \right.$

$$-dU_g = dW_g = V\, dQ = V^2 dC$$

$$dU_{tot} = dU_c + dU_g = \frac{1}{2} V^2 dC - V^2 dC = -\frac{1}{2} V^2 dC$$

$$F_x = -\frac{dU_{tot}}{dx} = \frac{1}{2} V^2 \frac{dC}{dx} = \frac{1}{2} V^2 \frac{\varepsilon_0 b}{d} (K-1)$$

$$E = V/d$$

$$\boxed{F_x = \frac{1}{2} E^2(x)\, \varepsilon_0 b d (K-1)}$$

$$E^2(x) = E^2 \text{ cost.} = \frac{V}{d}$$