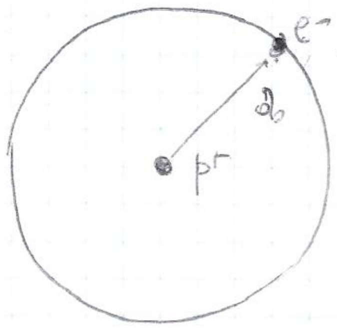
ES #6 Energia elettrostatica; operatori differenziali

01/XI/2020

(3-6)



* $V_0 = \frac{1}{4\pi\varepsilon_0} \frac{e}{a_0} = 27.17\,V$

$U_0 = -eV_0 = -27.17\,eV = -4.35 \cdot 10^{-18}\,J$

* $E_K = \frac{1}{2} m v^2$

$\sim \frac{mv^2}{r}$ $\quad m \frac{v^2}{a_0} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{a_0^2}$

$E_K = \frac{1}{2} m v^2 = \frac{1}{8\pi\varepsilon_0} \frac{e^2}{a_0} = 2{,}176 \cdot 10^{-18}\,J = 13{,}58\,eV$

$E_{tot} = E_K + U_0 = \frac{1}{8\pi\varepsilon_0} \frac{e^2}{a_0} - \frac{1}{4\pi\varepsilon_0} \frac{e^2}{a_0} = -\frac{1}{8\pi\varepsilon_0} \frac{e^2}{a_0} = -2{,}176 \cdot 10^{-18}\,J$
$= \underline{\underline{-13{,}6\,eV}}$

(3-7) $\quad \rho = 1.6\,\mu C/m^3$, $R = 10\,mm$ vacuum

$\frac{U}{V}$ $\quad u(\bar{r}) = \frac{\varepsilon_0}{2} \bar{E}_0^2(\bar{r}) = \frac{\varepsilon_0}{2} \bar{E}_0(\bar{r}) \cdot \bar{E}_0(\bar{r})$

$$U = \int_{\tau_\infty} u\, d\tau = \underbrace{\int_{\tau_\infty} \frac{\varepsilon_0 E_0^2}{2} d\tau}_{} = \frac{\varepsilon_0}{2} \int_{S=\partial\tau} V_0(\bar{r}) \bar{E}_0(\bar{r}) \cdot d\bar{S} + \frac{\varepsilon_0}{2} \int_{\tau} \bar{E}_0^2(\bar{r})\, d\tau$$

$$\bar{E}_0(\bar{r}) = \begin{cases} \frac{\rho}{3\varepsilon_0} r \, \hat{e}_r & r \leq R \\ \frac{\rho R^3}{3\varepsilon_0} \frac{1}{r^2} \hat{e}_r & r > R \end{cases}$$

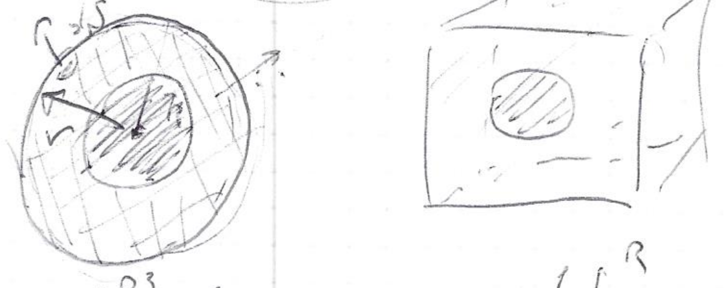
$$U = \int_{\tau_\infty} u \, d\tau = \frac{\varepsilon_0}{2} \left[\int_\emptyset^R \left(\frac{\rho r}{3\varepsilon_0}\right)^2 4\pi r^2 dr + \int_R^{+\infty} \left(\frac{\rho R^3}{3\varepsilon_0 r^2}\right)^2 4\pi r^2 dr \right] =$$

$$= \frac{\varepsilon_0}{2} \left[\frac{4\pi\rho^2}{9\varepsilon_0^2} \frac{1}{5} \left(R^5 - \emptyset\right) + \frac{4\pi \rho^2 R^6}{9\varepsilon_0} \left(-\emptyset + \frac{1}{R}\right) \right] =$$

$$= \frac{4\pi\rho^2 R^5}{15\varepsilon_0} = 1.51 \cdot 10^{-5} \, J$$

$$U = \frac{\varepsilon_0}{2} \oint_{S = \partial\tau} (V_0 \bar{E}_0) \cdot d\bar{S} + \frac{\varepsilon_0}{2} \int_\tau \bar{E}_0^2(\bar{r}) \, d\tau = \qquad V_0(\bar{r}) = \frac{\rho R^3}{3\varepsilon_0} \frac{1}{r}$$

$r > R$



$$= \frac{\varepsilon_0}{2} \underbrace{\frac{\rho R^3}{3\varepsilon_0} \frac{1}{r}}_{V_0(r)} \cdot \underbrace{\frac{\rho R^3}{3\varepsilon_0} \frac{1}{r^2}}_{E_{0r}(r)} \cdot \underbrace{4\pi r^2}_{dS(r)} + \frac{\varepsilon_0}{2} \left(\int_\emptyset^R + \int_R^r E_0^2(r') \, 4\pi r'^2 \, dr' \right) =$$

$$= \cancel{\frac{\varepsilon_0}{2} \frac{\rho R^6}{9\varepsilon_0} \frac{4\pi}{r}} + \frac{\varepsilon_0}{2} \left[\frac{1}{5} \frac{4\pi \rho^2 R^5}{9\varepsilon_0} + \frac{4\pi\rho^2 R^6}{9\varepsilon_0} \left(\frac{1}{R} - \cancel{\frac{1}{r}} \right) \right] =$$

$$= \frac{4\pi\rho^2 R^5}{15\varepsilon_0} = 1.51 \cdot 10^{-5} \, J$$

Gradiente

f

$$\boxed{df = \vec{\nabla} f \cdot \vec{dl}} \qquad \vec{\nabla} f = \frac{df}{\overline{dl}}\,\hat{dl}$$

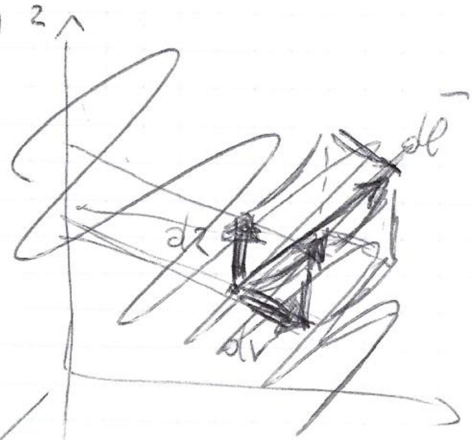
(A) $\vec{dl} = (dx, dy, dz)$

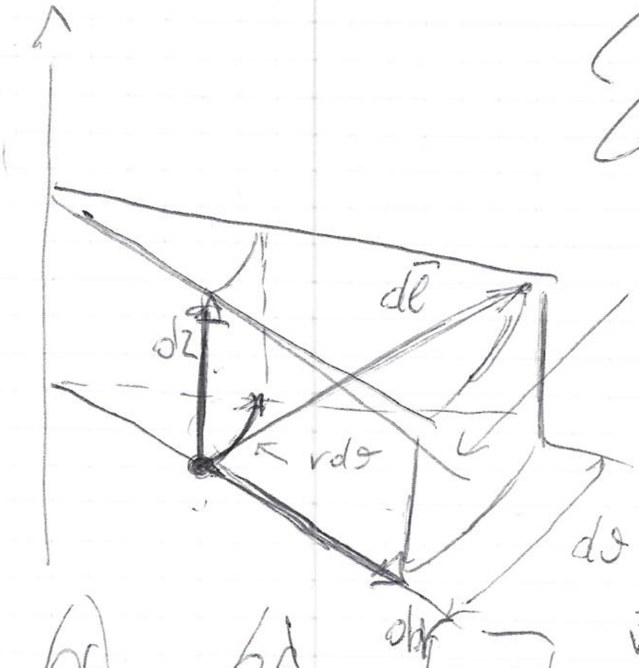
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\vec{\nabla} f)_x dx + (\vec{\nabla} f)_y dy + (\vec{\nabla} f)_z dz$$

$$\boxed{\vec{\nabla} f = \left(\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z}$$

(B) r, θ, z $\qquad \vec{dl} = (dr; r\,d\theta; dz) =$



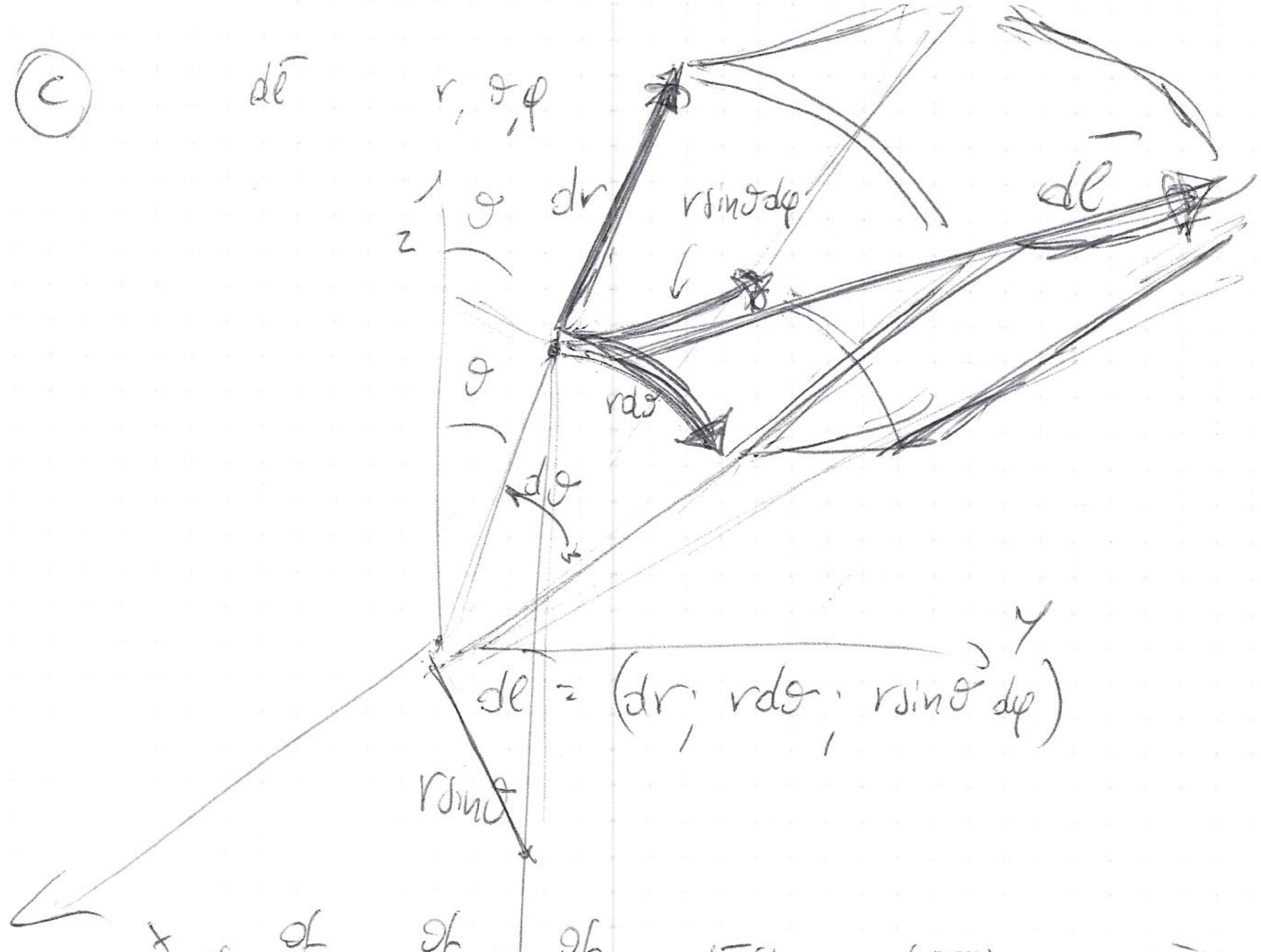


$\vec{\nabla} f \cdot \vec{dl}$

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial z}dz = (\vec{\nabla} f)_r dr + (\vec{\nabla} f)_\theta r\,d\theta + (\vec{\nabla} f)_z dz$$

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z}$$

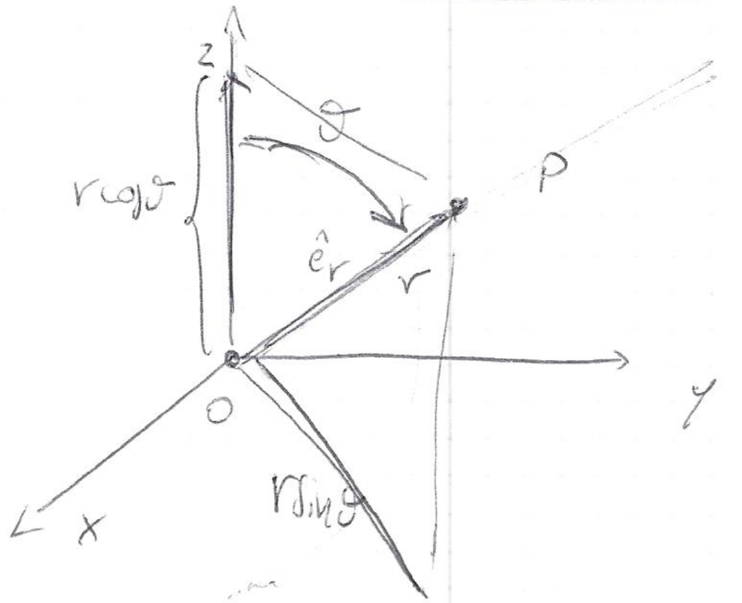
(c)

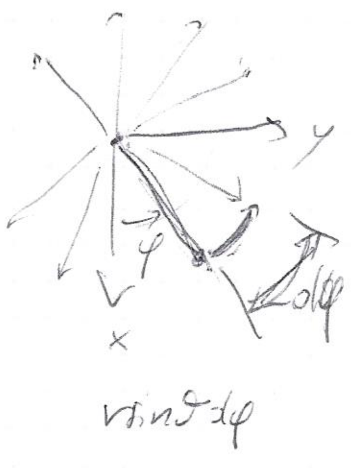


$$d\bar{l} = (dr;\ r d\vartheta;\ r\sin\vartheta\, d\varphi)$$

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{\partial f}{\partial \varphi} d\varphi = (\vec{\nabla} f)_r\, dr + (\vec{\nabla} f)_\vartheta\, r d\vartheta + (\vec{\nabla} f)_\varphi\, r\sin\vartheta\, d\varphi$$

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\hat{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi}$$





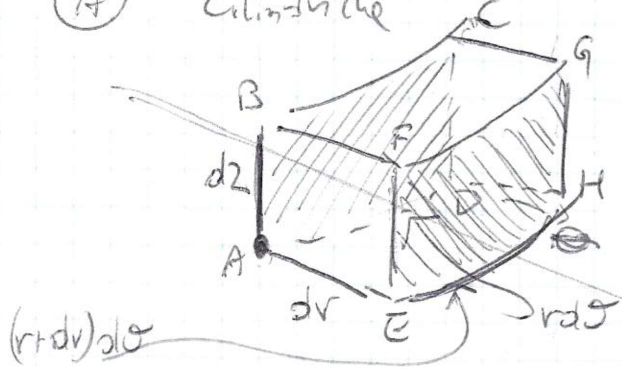
Divergenza

$$\Phi(\bar v) = \int_{S=\partial\tau} \bar v \cdot d\bar S = \int_\tau \operatorname{div} \bar v \, d\tau$$

$$\operatorname{div}(\bar v) = \nabla \cdot \bar v$$

$$d\Phi(\bar v) = \operatorname{div}(\bar v)\, d\tau$$

(A) Cilindriche



(r) $A(r,\theta,z)$

$$d\Phi_{ABCD} = \bar v(r,\theta,z)\cdot d\bar S = -v_r\, r\, d\theta\, dz$$

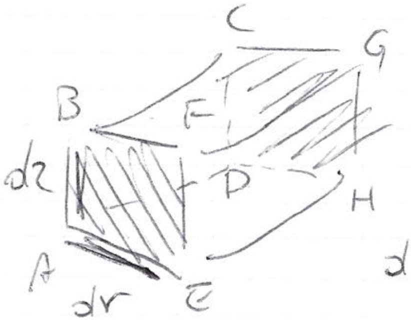
$$d\Phi_{EFGH} = \bar v(r+dr,\theta,z)\cdot d\bar S = \left[v_r(r,\theta,z) + \frac{\partial v_r}{\partial r} dr\right](r+dr)\, d\theta\, dz =$$

$$= v_r\, r\, d\theta\, dz + v_r\, dr\, d\theta\, dz + \frac{\partial v_r}{\partial r} r\, dr\, d\theta\, dz + \cancel{\frac{\partial v_r}{\partial r}(dr)^2 d\theta\, dz}$$

$$\frac{\partial}{\partial r}(r v_r) = v_r + r\frac{\partial v_r}{\partial r}$$

$$d\Phi_r = \frac{\partial}{\partial r}(r v_r)\, dr\, d\theta\, dz = \frac{1}{r}\frac{\partial}{\partial r}(r v_r)\underbrace{r\, dr\, d\theta\, dz}_{d\tau} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r)\, d\tau$$

(θ)



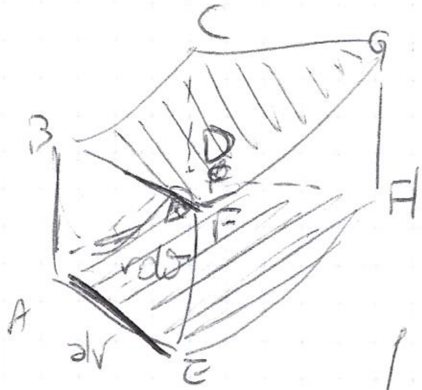
$$d\Phi_{ABFG} = \bar v(r,\theta,z)\cdot d\bar S = -v_\theta\, dr\, dz$$

$$d\Phi_{CDHG} = \bar v(r,\theta+d\theta,dz)\cdot d\bar S =$$

$$= \left[v_\theta(r,\theta,z) + \frac{\partial v_\theta}{\partial \theta} d\theta\right] dr\, dz = v_r\, dr\, dz + \frac{\partial v_\theta}{\partial \theta} dr\, d\theta\, dz$$

$$d\Phi_\theta = \frac{\partial v_\theta}{\partial \theta} dr\, d\theta\, dz = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\, d\tau$$

(2)



$$d\Phi_{ADHG} = -v_z(r,\vartheta,z)\, r\,dr\,d\vartheta$$

$$d\Phi_{BCGF} = \bar{J}(r,\vartheta,z+dz)\cdot d\bar{S} =$$

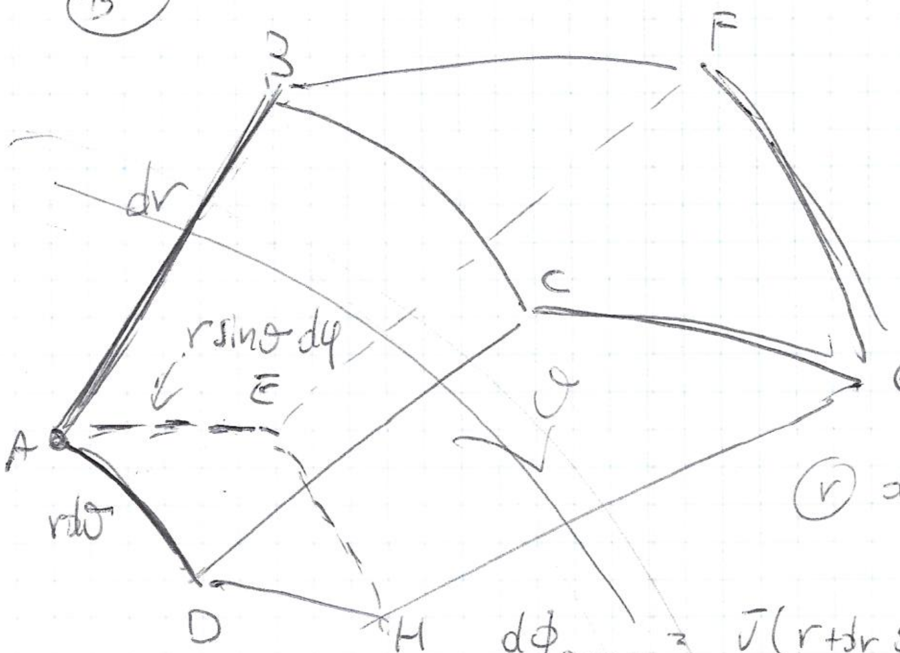
$$= \left[v_z(r,\vartheta,z) + \frac{\partial v_z}{\partial z} dz \right] r\,dr\,d\vartheta$$

$$d\Phi_z = \frac{\partial v_z}{\partial z} r\,dr\,d\vartheta\,dz = \frac{\partial v_z}{\partial z} d\tau$$

$$d\Phi_{tot} = d\Phi_r + d\Phi_\vartheta + d\Phi_z =$$

$$= \underbrace{\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{\partial v_z}{\partial z} \right]}_{\mathrm{div}(\bar{J})} d\tau$$

(B)



$A(r,\theta,\varphi)$

(r) $d\Phi_{ADHE} = -v_r\, r^2 \sin\theta\, d\theta\, d\varphi$

$d\Phi_{BFGC} = \vec{v}(r+dr,\theta,\varphi)\cdot d\vec{S} =$

$$= \left[v_r(r,\theta,\varphi) + \frac{\partial v_r}{\partial r} dr \right] (r+dr)^2 \sin\theta\, d\theta\, d\varphi =$$

$$= v_r r^2 \sin\theta\, d\theta\, d\varphi + 2 v_r r \sin\theta\, dr\, d\theta\, d\varphi + \cancel{v_r \sin\theta\, (dr)^2 d\theta\, d\varphi} +$$

$$+ \frac{\partial v_r}{\partial r} r^2 \sin\theta\, d\theta\, d\varphi + \cancel{\frac{\partial v_r}{\partial r} 2r \sin\theta\, (dr)^2 d\theta\, d\varphi} + \cancel{\frac{\partial v_r}{\partial r} \sin\theta\, (dr)^3 d\theta\, d\varphi}$$

$$\longrightarrow \frac{\partial}{\partial r}(r^2 v_r) \quad \Rightarrow d\Phi_r = \frac{\partial}{\partial r}(r^2 v_r) \sin\theta\, dr\, d\theta\, d\varphi$$

$$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$d\Phi_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)\, d\tau$$

(θ) $d\Phi_{ABFE} = \vec{v}(r,\theta,\varphi)\cdot d\vec{S} = -v_\theta\, r \sin\theta\, dr\, d\varphi$

$$d\Phi_{DCGH} = \vec{v}(r,\theta+d\theta,\varphi)\cdot d\vec{S} = \left[v_\theta(r,\theta,\varphi) + \frac{\partial v_\theta}{\partial \theta} d\theta \right] r \underbrace{\sin(\theta+d\theta)}\, dr\, d\varphi =$$

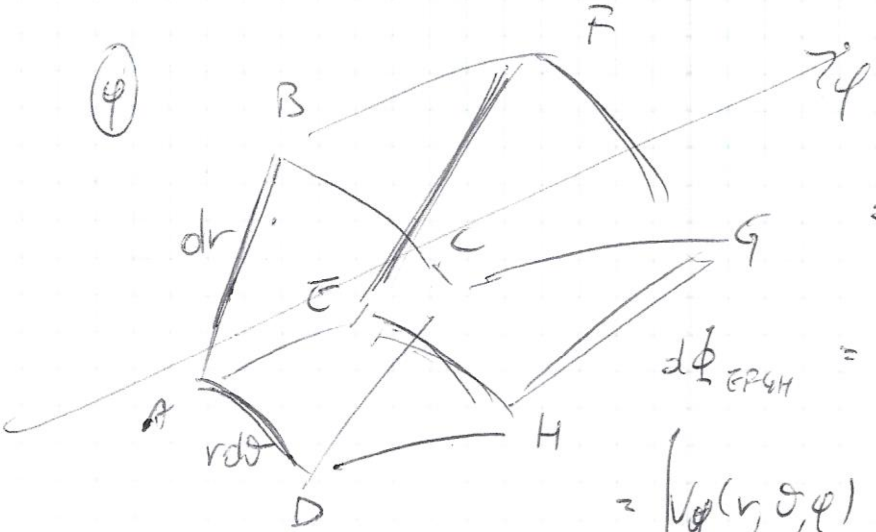
$$\left[\sin(\theta+d\theta) = \sin\theta + \frac{\partial}{\partial\theta}(\sin\theta)\, d\theta = \sin\theta + \cos\theta\, d\theta \right]$$

$$= v_\theta r \sin\theta\, dr\, d\varphi + v_\theta r \cos\theta\, dr\, d\theta\, d\varphi + \frac{\partial v_\theta}{\partial \theta} r \sin\theta\, dr\, d\theta\, d\varphi + \cancel{\frac{\partial v_\theta}{\partial \theta} r \cos\theta\, dr\, (d\theta)^2 d\varphi}$$

$$\frac{\partial}{\partial\vartheta}(\sin\vartheta\, v_\vartheta) = v_\vartheta\cos\vartheta + \sin\vartheta\frac{\partial v_\vartheta}{\partial\vartheta}$$

$$d\Phi_\vartheta = \frac{\partial}{\partial\vartheta}(\sin\vartheta\cdot v_\vartheta)\, r\,dr\,d\vartheta\,d\varphi = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, v_\vartheta)\,d\tau$$

φ



$$d\Phi_{ABCD} = -v_\varphi\, r\,dr\,d\vartheta$$

$$d\Phi_{EFGH} = \vec{v}(r,\vartheta,\varphi+d\varphi)\cdot d\vec{S} =$$

$$= \left[v_\varphi(r,\vartheta,\varphi) + \frac{\partial v_\varphi}{\partial\varphi}d\varphi\right] r\,dr\,d\vartheta$$

$$d\Phi_\varphi = \frac{\partial v_\varphi}{\partial\varphi}\, r\,dr\,d\vartheta\,d\varphi = \underline{\underline{\frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial\varphi}}}\,d\tau$$

$$d\Phi_{Tot} = \underbrace{\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(v_\vartheta\sin\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial\varphi}\right]}_{\mathrm{div}(\vec{v})}d\tau$$

Coord. curvilinee ortogonali (q_1, q_2, q_3)

$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$

Gradiente $df = \bar{\nabla} f \cdot d\bar{l}$

Coord. q_1 $\hat{e}_i = \frac{1}{h_i} \frac{\partial \bar{r}}{\partial q_i}$ $h_i = \left| \frac{\partial \bar{r}}{\partial q_i} \right|$

spost. infinitesimo lungo q_1 $dl_1 = h_1(q_1, q_2, q_3) dq_1$

$$d\bar{l} = (dl_1, dl_2, dl_3) = dl_1 \hat{e}_1 + dl_2 \hat{e}_2 + dl_3 \hat{e}_3 =$$

$$= h_1 \, dq_1 \hat{e}_1 + h_2 \, dq_2 \hat{e}_2 + h_3 \, dq_3 \hat{e}_3$$

Cart.: $h_1 = h_2 = h_3 = 1$

Cyl.: $h_1 = 1$; $h_2 = r$; $h_3 = 1$

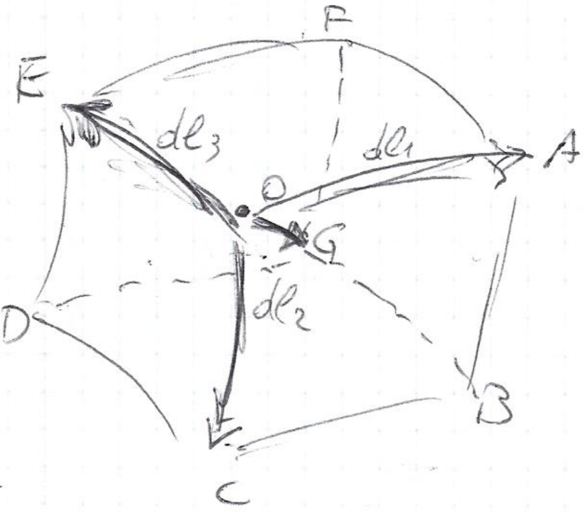
dr $\quad r d\vartheta$ $\quad dz$

Sph.: $h_1 = 1$; $h_2 = r$; $h_3 = r \sin\vartheta$

$$df = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 = (\bar{\nabla} f)_1 h_1 dq_1 + (\bar{\nabla} f)_2 h_2 dq_2 + (\bar{\nabla} f)_3 h_3 dq_3$$

$$\boxed{\bar{\nabla} f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial q_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial q_3} \hat{e}_3}$$

Divergenza



$d\Phi_1 =$

$$d\Phi_{OCDE} = \bar{v}(O)\cdot d\bar{S} =$$

$$= -v_1(O)\, dl_2(O)\, dl_3(O) = -v_1(O)\, h_1(O)\, h_2(O)\, dq_2\, dq_3$$

$$d\Phi_{ABGF} = \bar{v}(A)\cdot d\bar{S} = v_1(A)\, h_2(A)\, h_3(A)\, dq_2\, dq_3 =$$

$$= \left[(v_1 h_2 h_3)_0 + \frac{\partial}{\partial q_1}(v_1 h_2 h_3)_0\, dq_1 \right] dq_2\, dq_3$$

$$d\Phi_1 = \frac{\partial}{\partial q_1}(v_1 h_2 h_3)\, dq_1\, dq_2\, dq_3 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q_1}(v_1 h_2 h_3)\, d\tau$$

$$d\tau = dl_1\, dl_2\, dl_3 = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$

$$d\Phi_2 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q_2}(v_2 h_1 h_3)\, d\tau$$

$$d\Phi_3 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q_3}(v_3 h_1 h_2)\, d\tau$$

$$d\Phi_{tot} = \underbrace{\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(v_1 h_2 h_3) + \frac{\partial}{\partial q_2}(v_2 h_1 h_3) + \frac{\partial}{\partial q_3}(v_3 h_1 h_2)\right]}_{\mathrm{div}(\bar{v})} d\tau$$

$$\bar{\nabla}\cdot\bar{E} = -\rho/\varepsilon_0$$

$$\bar{E}_0 = -\bar{\nabla}V_0 \qquad -\bar{\nabla}\cdot(\nabla V_0) = -\rho/\varepsilon_0$$

$$\boxed{\begin{aligned}\nabla^2 V_0 &= -\rho/\varepsilon_0\\ \Delta V_0 &= -\rho/\varepsilon_0\end{aligned}} \qquad \boxed{\nabla^2 V_0 = \Delta V_0 = -\frac{\rho}{\varepsilon_0}}$$

eq. di Poisson

$$\nabla^2 V_0 = \phi \qquad \text{eq. di Laplace}$$

$$\bar{u} = \bar{\nabla}f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z = u_x\hat{e}_x + u_y\hat{e}_y + u_z\hat{e}_z$$

$$\nabla\cdot\bar{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\bar{u} = \bar{\nabla}f = \underbrace{\frac{\partial f}{\partial r}}_{u_r}\hat{e}_r + \underbrace{\frac{1}{r}\frac{\partial f}{\partial \vartheta}}_{u_\vartheta}\hat{e}_\vartheta + \underbrace{\frac{\partial f}{\partial z}}_{u_z}\hat{e}_z$$

$$\nabla\cdot\bar{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{\partial u_z}{\partial z} =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \vartheta}\left(\frac{1}{r}\frac{\partial f}{\partial \vartheta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \vartheta^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

$$\nabla^2\bar{E} = \nabla^2(E_x\hat{e}_x + E_y\hat{e}_y + E_z\hat{e}_z)$$