ES. #7 I eq. di Maxwell; Laplace/Poisson; conduttori — 11/XII/2020

(1)

$$\bar\nabla\cdot\bar{E} = \rho/\varepsilon_0 \qquad \bar{E} = -\bar\nabla V_0$$

$$\nabla^2 V_0 = -\rho/\varepsilon_0 \quad \text{eq. di Poisson}$$

quando $\rho = \phi$ $\nabla^2 V_0 = \phi$ eq. di Laplace

+ b.c. (boundary conditions)

Laplace $\nabla^2 V_0 = \phi$.

Teorema di Earnshaw (1842)

∄ eq. stabile di cariche ptformi solo con $\bar{E}_0$

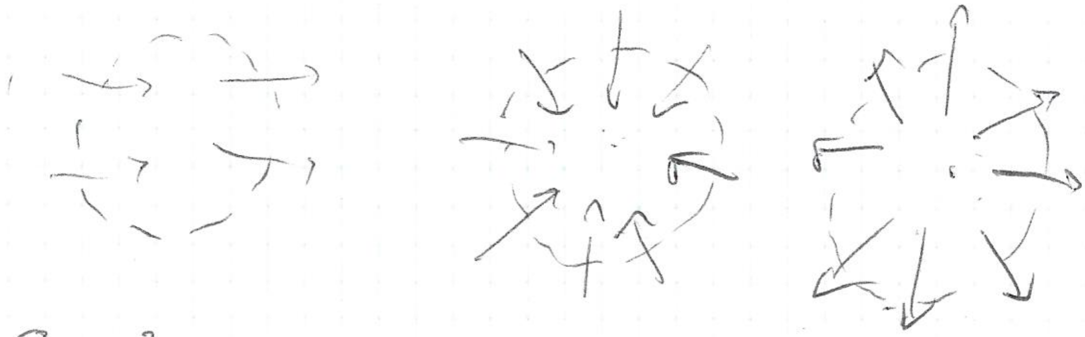
Eq. stabile: ∀ perturbazione c'è forza di ritorno → equilibrio

↳ energeticamente: minimo di energia

Cons.

$\bar\nabla\cdot\bar{E} = \phi$

$\phi(\bar{E}_s) = \phi$



$\nabla^2 V_0 = \phi$

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial z^2} = \phi$$

patatina Pringle

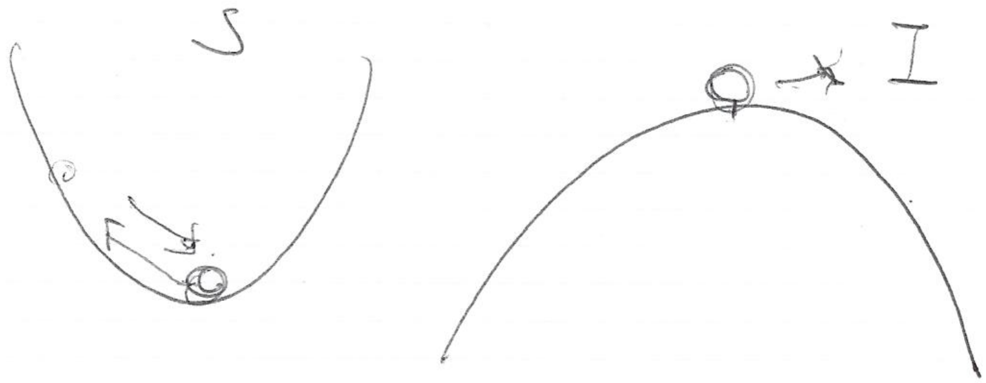
f. a sella

Conduttori in elettrostatica : $\nexists\ \bar{E}_0$ all'interno (2)

↳ equipotenziali → b.c. di Dirichlet

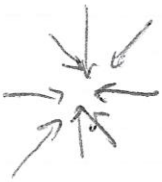
→ q superficie dei cond. → b.c. $\propto E_{0n} = -\dfrac{\partial U_0}{\partial x_n}$

b.c. di Neumann



Quadrupolo elettrostatico (dimensionamento)

pt di eq. $O(0,0)$ → $\bar{E}_0(x,y)$ $\begin{cases} E_{0x} = -Ax \\ E_{0y} = -Ay \end{cases}$



$$\bar{\nabla}\cdot\bar{E}_0 = \frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} = -A - A = -2A \neq 0$$

$$\begin{cases} E_{0x} = -Ax \\ E_{0y} = Ay \end{cases} \qquad \nabla\cdot\bar{E}_0 = -A + A = 0$$

$$\frac{\partial U_0}{\partial x} = -E_{0x} \rightsquigarrow U_0(x,y) = -\int E_{0x}\,dx + f(y)$$

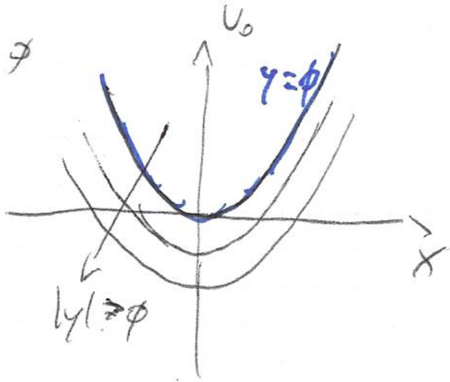
$$U_0 = \frac{A}{2}x^2 + f(y) \quad ; \quad \frac{\partial U_0}{\partial y} = -E_{0y} \Rightarrow \frac{\partial f(y)}{\partial y} = -E_{0y} = -Ay$$

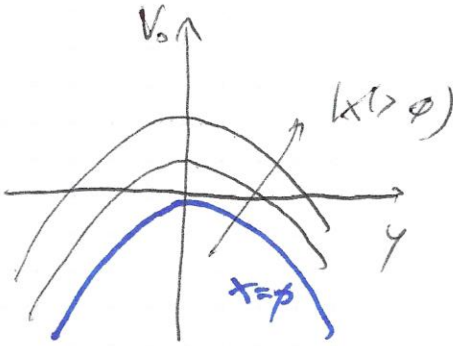
$f(y) = -\int E_{0y}\,dy + C = -\frac{A}{2}y^2 + C$ 3

$V_0(x,y) = \frac{A}{2}(x^2-y^2) + C$ $C=\phi$

$V_0(x,y) = \frac{A}{2}(x^2-y^2)$ QUADRUPOLO o punti a sella

$\nabla^2 V_0 = \phi$





Cond.: sup. equipot.

$$V_0 = \frac{A}{2}(x^2-y^2) = K$$

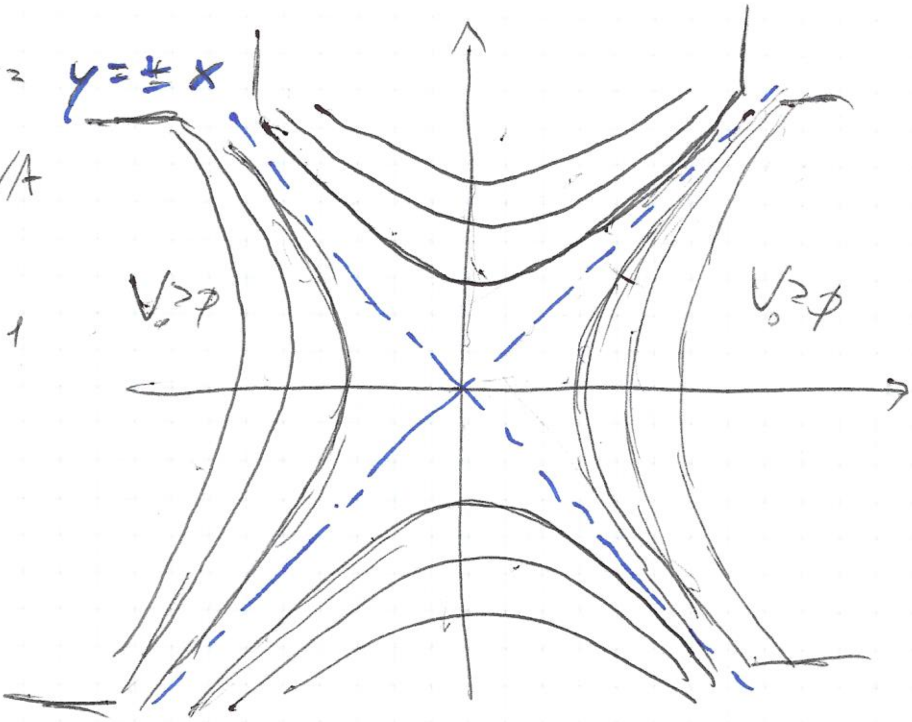
$\boxed{K=\phi}$ $x^2=y^2 \Rightarrow y=\pm x$

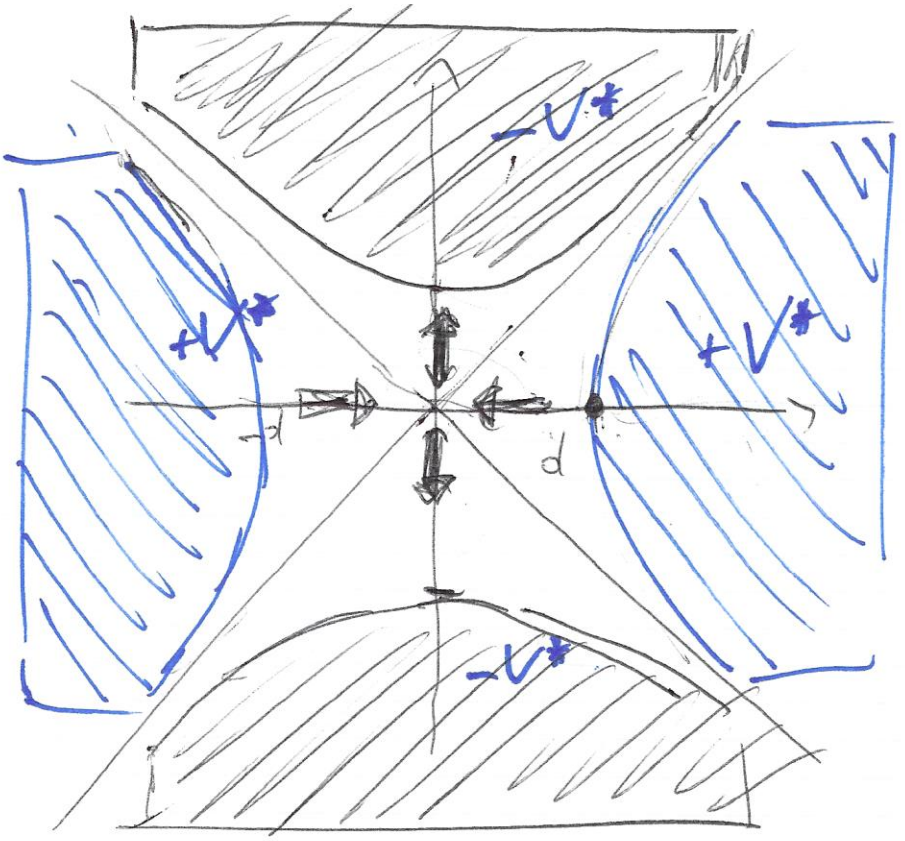
$\boxed{K>\phi}$ $a^2 = 2K/A$

iperboli $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$

$\boxed{K<\phi}$ $a^2 = -2K/A$

iperboli $\frac{x^2}{a^2} - \frac{y^2}{a^2} = -1$





$$V_0(x,y) = \frac{A}{2}(x^2-y^2)$$

$$\frac{A}{2} d^2 = V^*$$

$$A = 2V^*/d^2$$

$$V_0(x,y) = \frac{V^*}{d^2}(x^2-y^2)$$

$$\bar{E}_0 = -\bar{\nabla} V_0$$

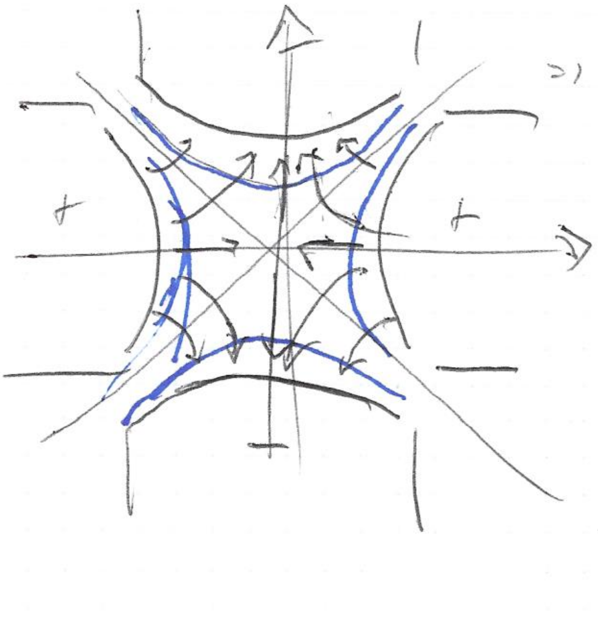
Linee di campo $\forall$pt, tg $\bar{E}_0$ $\quad d\bar{l} \parallel \bar{E}_0$

$$\frac{dx}{E_{0x}} = \frac{dy}{E_{0y}} \leftarrow E_{0x} = -Ax; \quad E_{0y} = Ay$$

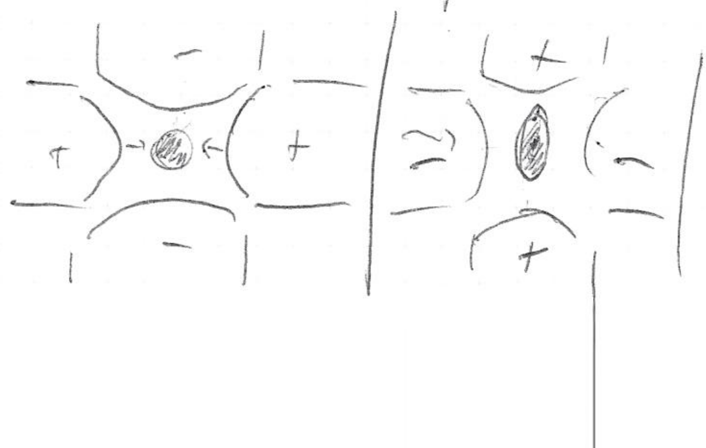
$$-\frac{dx}{x} = \frac{dy}{y}$$

$$-\log x + \log k = \log y$$

$$\Rightarrow \log(xy/k) = 0 \qquad xy = k \qquad y = k/x$$



Fascio di particelle cariche



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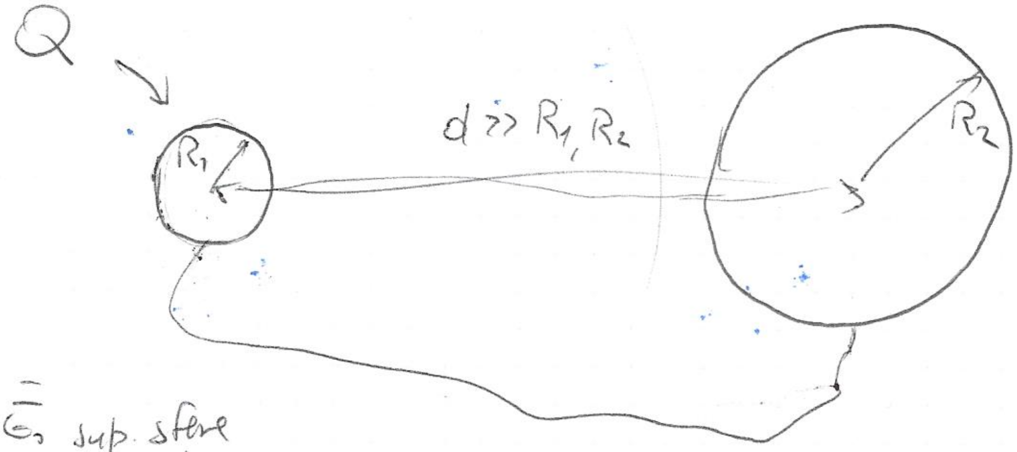


filo trascurabile

$\bar{E}_0$ sup. sfere

$Q_1, Q_2 \,/\, Q = Q_1 + Q_2$

$$V_0 = V_{01} = V_{02}$$

$d \gg R_1, R_2$ ignoriamo interazione

$\Rightarrow Q_1, Q_2$ uniformi

$$V_0 = V_{01} \; \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{R_1} \qquad \frac{Q_1}{r}$$

$$V_0 = V_{02} = \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{R_2}$$

$$Q = Q_1 + Q_2 = 4\pi\varepsilon_0 V_0 (R_1 + R_2)$$

$$V_0 = \frac{Q}{4\pi\varepsilon_0 (R_1 + R_2)}$$

$$Q_1 = \frac{R_1}{R_1 + R_2} Q$$

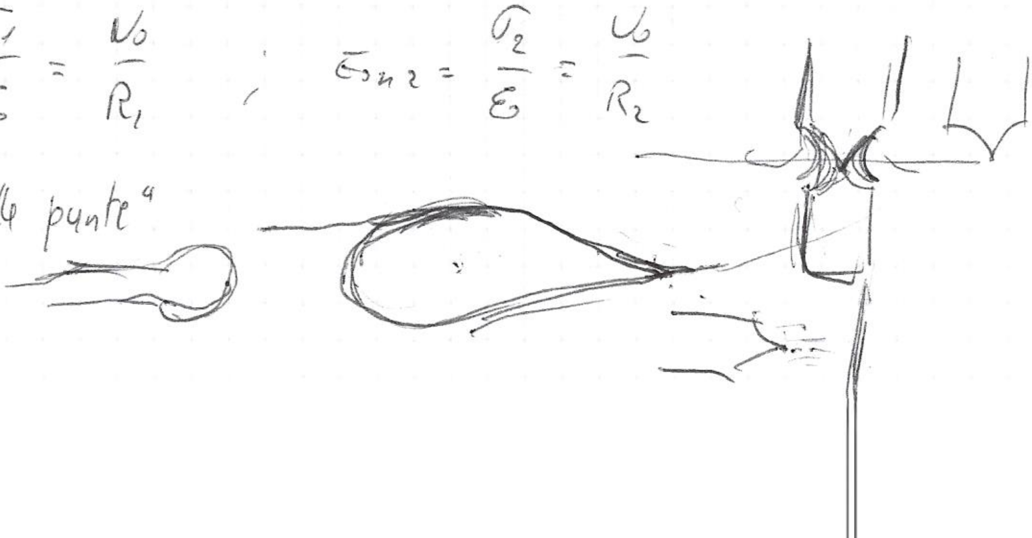
$$Q_2 = \frac{R_2}{R_1 + R_2} Q$$

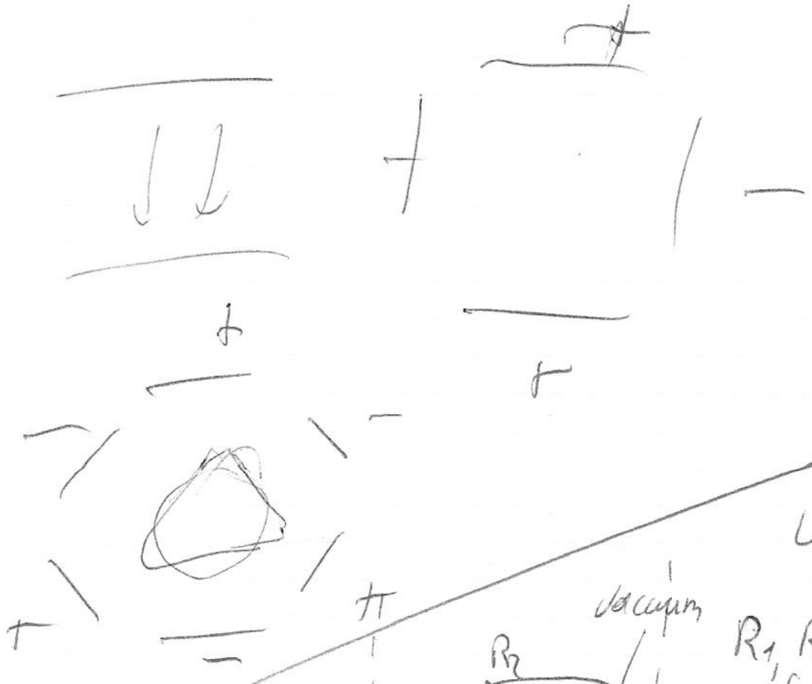
$\bar{E}_0 = \frac{\sigma}{\varepsilon}\hat{e}_n$ $\quad \sigma$ carica superf. sul conduttore

$$\sigma_1 = \frac{Q_1}{4\pi R_1^2} = \frac{\varepsilon_0 V_0}{R_1} \; ; \quad \sigma_2 = \frac{Q_2}{4\pi R_2^2} = \frac{\varepsilon_0 V_0}{R_2}$$

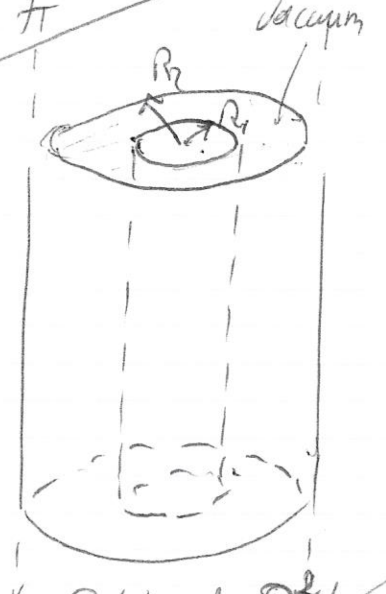
$$E_{0n1} = \frac{\sigma_1}{\varepsilon_0} = \frac{V_0}{R_1} \; ; \quad E_{0n2} = \frac{\sigma_2}{\varepsilon_0} = \frac{V_0}{R_2}$$

"potere delle punte"





Laplace = geom. cil.



R_1, R_2

$$\begin{cases} \nabla^2 V_0 = 0 \\ V_0(R_1) = V_1 \\ V_0(R_2) = V_2 \quad \leftarrow \text{b.c.} \\ R_1 < r < R_2 \end{cases}$$

$$\nabla^2 V_0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_0}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 V_0}{\partial \vartheta^2} + \frac{\partial^2 V_0}{\partial z^2}} = 0$$

ϑ, z indipendent

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_0}{dr}\right) = 0 \hookrightarrow r\frac{dV_0}{dr} = C_1$$

$$\frac{dV_0}{dr} = \frac{C_1}{r} \qquad dV_0 = C_1\frac{dr}{r} \rightsquigarrow \underline{V_0(r) = C_1 \log r + C_2}$$

$$V_0(R_1) = V_1 = C_1 \log R_1 + C_2$$

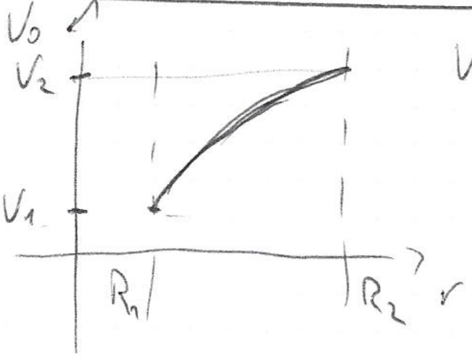
$$V_0(R_2) = V_2 = C_1 \log R_2 + C_2$$

$$V_1 - V_2 = C_1 \log(R_1/R_2) \rightsquigarrow \underline{C_1} = \frac{V_1 - V_2}{\log(R_1/R_2)}$$

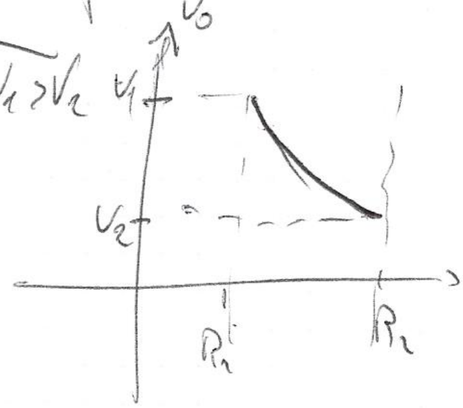
$$\underline{C_2} = V_2 - \frac{V_1 - V_2}{\log(R_1/R_2)}\log R_2$$

$$\boxed{V_0(r) = \frac{V_1 - V_2}{\log(R_1/R_2)} \log\left(\frac{r}{R_2}\right) + V_2}$$

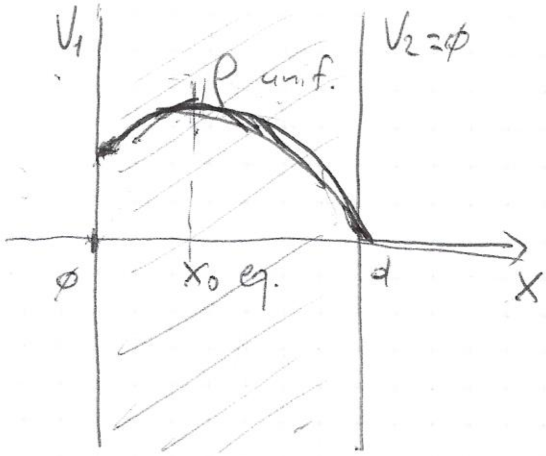
$V_1 < V_2$



$V_1 > V_2$



Poisson - cartesiano



$\nabla^2 V_0 \rightsquigarrow$

$$\begin{cases} \dfrac{d^2 V_0}{dx^2} = -\dfrac{\rho}{\varepsilon_0} \\ V_0(\phi) = V_1 \\ V_0(d) = \phi \end{cases}$$

$$\frac{dV_0}{dx} = -\frac{\rho}{\varepsilon_0} x + C_1$$

$$V_0(x) = -\frac{\rho}{2\varepsilon_0} x^2 + C_1 x + C_2$$

b.c.: $V_0(\phi) = \boxed{C_2 = V_1}$

$\underline{V_0(d)} = -\dfrac{\rho d^2}{2\varepsilon} + C_1 d + V_1 = \phi \quad \rightsquigarrow C_1 = \dfrac{\rho d}{2\varepsilon_0} - \dfrac{V_1}{d}$

$$\boxed{V_0(x) = -\frac{\rho}{2\varepsilon_0} x^2 + \left(\frac{\rho d}{2\varepsilon_0} - \frac{V_1}{d}\right) x + V_1}$$

$E_{0x} = \phi \iff \left.\dfrac{dV_0}{dx}\right|_{x_0} = \phi \quad -\dfrac{\rho}{\varepsilon_0} x_0 + \dfrac{\rho d}{2\varepsilon_0} - \dfrac{V_1}{d} = \phi$

$$\rightsquigarrow x_0 = \frac{d}{2} - \frac{\varepsilon_0 V_1}{\rho d}$$

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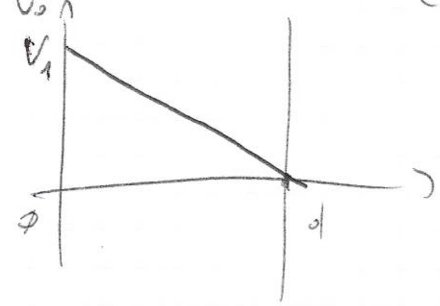
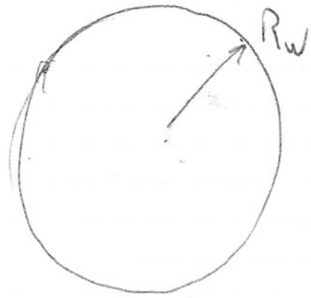
$\rho = \phi$ $\quad V_0(x) = V_1\left(1 - \dfrac{x}{d}\right)$



$\left(\text{Laplace} \quad \nabla^2 V_0 = \phi\right)$

Separazione delle variabili in Laplace



2-D

$\nabla^2 V_a(r,\vartheta) = \phi$

$V_a(R_W, \vartheta) = f(\vartheta)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \vartheta^2} = \phi$$

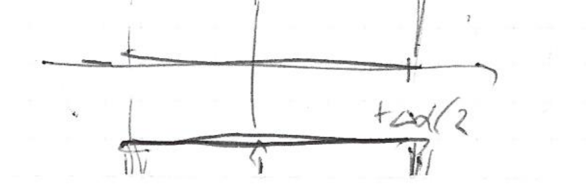
b.c. 1 opp. 2 $\longrightarrow V(R_W, \vartheta) = V_0 \cos\vartheta$

$\downarrow$

$V(R_W, \vartheta) = V_0\left[H(\vartheta + \Delta\alpha/2) - H(\vartheta - \Delta\alpha/2)\right]$

gradino $H(x - x_0)\begin{cases} 1 & x > x_0 \\ \phi & x < x_0 \end{cases}$

(step) Heaviside



$\Delta\alpha$

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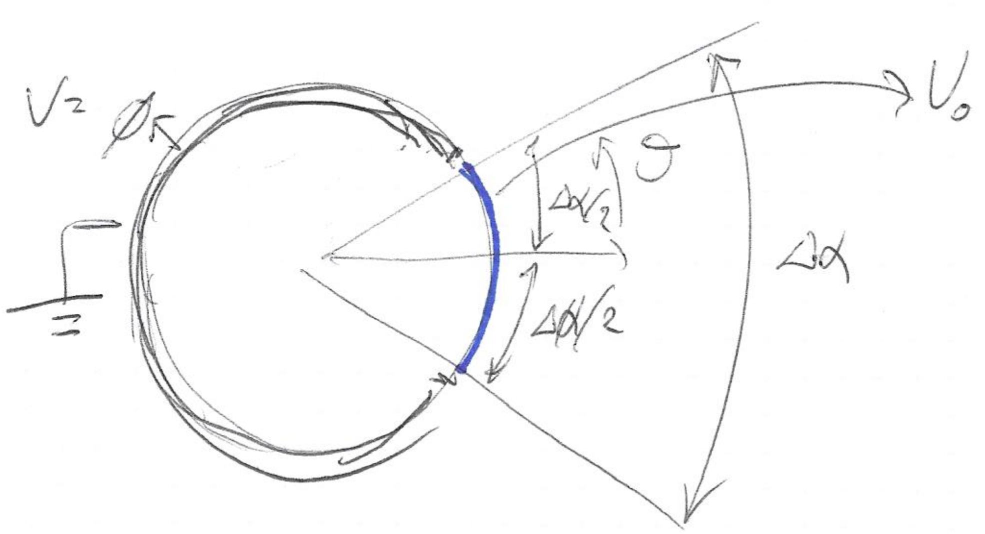

$$V(r,\theta) = R(r)\cdot T(\theta)$$

$$\frac{T(\theta)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{R(r)}{r}\frac{\partial^2 T(\theta)}{\partial \theta^2} = \phi \qquad \times \frac{r^2}{R(r)T(\theta)}$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{T}\frac{\partial^2 T}{\partial \theta^2}$$

$\downarrow$

entrambi = una qualche costante

(A) $\frac{1}{T}\frac{\partial^2 T}{\partial \theta^2} = -m^2$ $\qquad T(\theta) = A_m \cos(m\theta) + \cancel{B_m \sin(m\theta)}$

$T(-\theta) = T(\theta) \qquad B_m = \phi$

$T(\theta) = A_m \cos(m\theta)$

(B) $\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = m^2 \rightarrow \underbrace{r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - m^2 R = \phi}_{\text{eq. di Cauchy-Eulero}}$

$\sim r^m \qquad m$ interi

$\sim \alpha_m r^m + \beta_m r^{-m} \qquad m = 0, 1, 2, \ldots$

$r \in [0, R_w] \qquad \beta_m = \phi$

$$\boxed{V(r,\vartheta) = R(r)\,T(\vartheta) = A_0 + \sum_{m=1}^{+\infty} A_m r^m \cos(m\vartheta)}$$

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+ b.c. $r = R_W$

= sol. autentica sol. $\Rightarrow$ A_m

($V_0 \cos\vartheta$ $A_m \neq 0$ $m=1$)