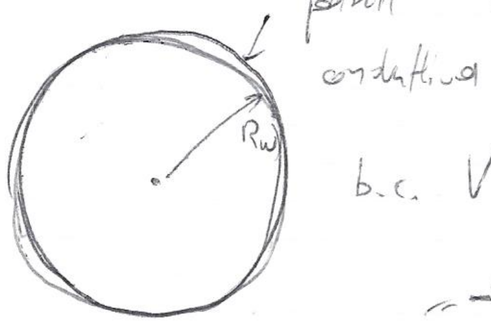
ES. # 8 13/XI/2020

5-6 Laplace - Poisson - conduttori ①

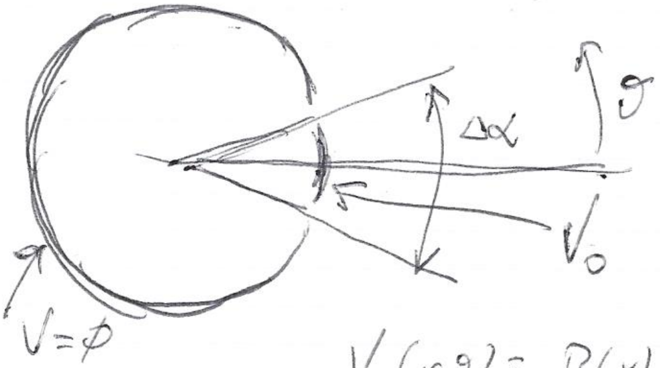


b.c. $V(r = R_W, \vartheta) = f(\vartheta)$

$\Delta\vartheta \to \phi$ $f(\vartheta)$ continua

$V(R_W, \vartheta) = V_0 \cos\vartheta$

$V(R_W, \vartheta) = V_0 \left[H(\vartheta + \Delta\alpha/2) - H(\vartheta - \Delta\alpha/2) \right]$



$V(R_W, \vartheta) = V(R_W, -\vartheta)$

$V(r, \vartheta) = R(r)\, T(\vartheta)$

$\nabla^2 V = \phi$

$$\frac{r}{R} \frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r} \right) = - \frac{1}{T} \frac{\partial^2 T}{\partial \vartheta^2}$$

Ⓐ $\frac{1}{T} \frac{\partial^2 T}{\partial \vartheta^2} = - m^2$ $T(\vartheta) = A_m \cos(m\vartheta) + \cancel{B_m \sin(m\vartheta)}$

Ⓑ $\boxed{r \frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r} \right) - m^2 R = \phi}$ Cauchy - Eulero

$\alpha_m r^m + \underline{\beta_m r^{-m}}$ $m \in \mathbb{Z}^+$

$\beta_m = \phi$

$$V(r,\vartheta) = R(r)T(\vartheta) = A_0 + \sum_{m=1}^{+\infty} A_m r^m \cos(m\vartheta)$$ (2)

(1) $V(R_W,\vartheta) = V_0 \cos\vartheta$

$r = R_W$

$$V(R_W,\vartheta) = A_0 + \sum_{m=1}^{+\infty} A_m R_W^m \cos(m\vartheta) = A_0 + A_1 R_W \cos\vartheta + A_2 R_W^2 \cos(2\vartheta) + \dots$$

$\to = V_0 \cos\vartheta$

$A_0 = \emptyset$

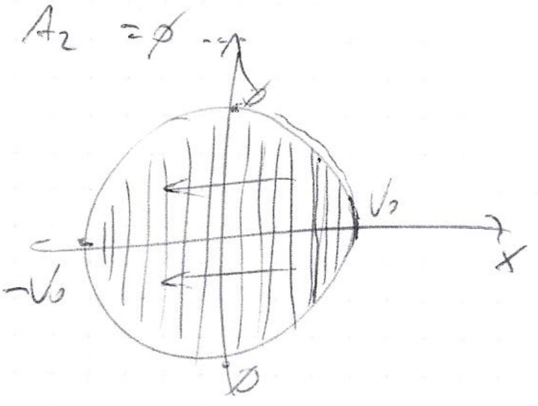
$A_1 R_W = V_0 \Rightarrow A_1 = V_0/R_W$

$A_2 = \emptyset \dots$

$$V(r,\vartheta) = V_0 \frac{r}{R_W}\cos\vartheta = \frac{V_0}{R_W} x$$

$$\bar{E}_0 = -\bar{\nabla} V = -\frac{\partial V_1}{\partial x}\hat{e}_x \cancel{- \frac{\partial V_1}{\partial y}\hat{e}_y - \frac{\partial V_1}{\partial z}\hat{e}_z}$$

$$= -\frac{V_0}{R_W}\hat{e}_x$$



(2)

$$V_0\left[H(\vartheta+\Delta\alpha/2) - H(\vartheta-\Delta\alpha/2)\right] = A_0 + \sum_{m=1}^{+\infty} A_m R_W^m \cos(m\vartheta)$$

$$\int_{-\pi}^{\pi} \cos(n\vartheta)\, d\vartheta \qquad \int_{-\pi}^{\pi} \cos(n\vartheta)\, d\vartheta$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\,\delta_{mn}$$ → delta di Kronecker

$$\delta_{mn}\begin{cases}1 & m=n\\ \emptyset & m\neq n\end{cases}$$

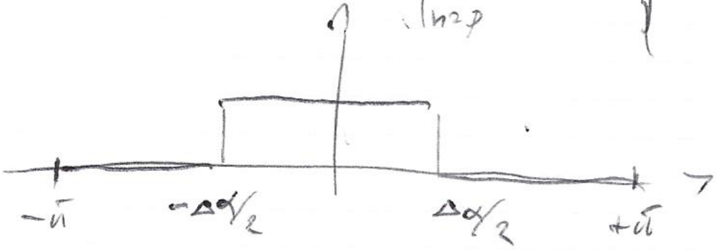
$n = \emptyset$

$n \neq \emptyset$

$\boxed{n=0}$ $$\int_{-\pi}^{\pi} V_0 \left[H(\theta+\Delta\alpha/2) - H(\theta-\Delta\alpha/2)\right] \cdot 1 \, d\theta = \int_{-\pi}^{\pi} A_0 \, d\theta = 2\pi A_0$$

$$\cos(n\theta)\Big|_{n=0} = 1$$

$$\int_{-\pi}^{\pi} 1 \cdot A_m R_w^m \cos(m\theta) \, d\theta$$



$$\int_{-\Delta\alpha/2}^{+\Delta\alpha/2} V_0 \, d\theta = V_0 \Delta\alpha \rightarrow V_0 \Delta\alpha = 2\pi A_0$$

$$\Rightarrow \boxed{A_0 = V_0 \frac{\Delta\alpha}{2\pi}}$$

$\boxed{n \neq 0}$

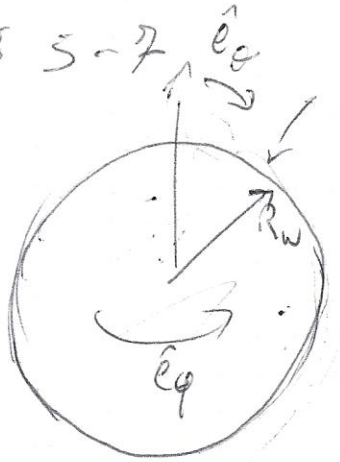
I membro $$\int_{-\Delta\alpha/2}^{\Delta\alpha/2} V_0 \cos(n\theta) \, d\theta = V_0 \left[\frac{\sin(n\theta)}{n}\right]_{-\Delta\alpha/2}^{\Delta\alpha/2} = \frac{2V_0}{n} \sin\left(\frac{n\Delta\alpha}{2}\right)$$

II membro $$A_n R_w^n \int_{-\pi}^{\pi} \cos^2(n\theta) \, d\theta = \pi A_n R_w^n$$

$$\left[\frac{1}{T} \int \cos^2(\omega t) \, dt = \frac{1}{2}\right]$$

$$\boxed{A_n = \frac{2V_0}{\pi} \frac{1}{R_w^n} \sin\left(\frac{n\Delta\alpha}{2}\right)}$$

$$V(r,\theta) = \frac{V_0 \Delta\alpha}{2\pi} + \sum_{m=1}^{+\infty} \frac{2V_0}{\pi m} \sin\left(\frac{m\Delta\alpha}{2}\right) \left(\frac{r}{R_w}\right)^m \cos(m\theta)$$



$$\nabla^2 V = \phi$$

$$\sigma_0(R_W, \theta) = K\cos\theta$$

$$\nabla^2_{sph} V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}} = \phi$$

$$V(r,\theta) = R(r)\,T(\theta)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)$$

$$\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = -\ell(\ell+1)$$

sol. nota $T(\theta) = P_\ell(\cos\theta)$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \quad \text{formula di Rodrigues}$$

$P_0(x) = 1$ $\qquad P_0(\cos\theta) = 1$

$P_1(x) = x$ $\qquad P_1(\cos\theta) = \cos\theta$

$P_2(x) = \frac{1}{2}(3x^2-1)$ $\qquad \dots$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1) \rightsquigarrow \text{sol. nota } R(r) = Ar^\ell + B\frac{1}{r^{\ell+1}}$$

$$V(r,\theta) = \sum_{\ell=\phi}^{+\infty}\left(A_\ell r^\ell + \frac{B^\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

I $\quad r \le R \quad \lim_{r\to\phi} B^\ell/r^{\ell+1} \Rightarrow \infty \qquad B_\ell = \phi$

II $\quad r \ge R \quad \lim_{r\to\infty} A_\ell r^\ell = \infty \qquad A_\ell = \phi$

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$$\begin{cases} V^{I}(r,\vartheta) = \sum_{\ell=\phi}^{+\infty} A_\ell r^\ell P_\ell(\cos\vartheta) & r\leq R \\ V^{II}(r,\vartheta) = \sum_{\ell=\phi}^{+\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\vartheta) & r\geq R \end{cases}$$

1. Potenziale e' continuo $\quad V^{I}(R_W,\vartheta) = V^{II}(R_W,\vartheta)$

$$\sum_\ell \boxed{A_\ell R_W^\ell} P_\ell(\cos\vartheta) = \sum_\ell \boxed{B_\ell / R_W^{\ell+1}} P_\ell(\cos\vartheta)$$

$$B_\ell = A_\ell R_W^{2\ell+1}$$

2. $\hat{e}_n = \hat{e}_r \quad \underset{(R_W^+,\vartheta)}{E_{0r}^{ext}} - \underset{(R_W^-,\vartheta)}{E_{0r}^{int}} = \frac{\sigma_0(R_W,\vartheta)}{\varepsilon_0} \qquad E_{0r} = -\frac{\partial V}{\partial r}$

$$\left[\underbrace{\frac{\partial V^{II}}{\partial r}}_{-E_{0r}^{ext}} - \underbrace{\frac{\partial V^{I}}{\partial r}}_{+E_{0r}^{int}}\right]_{r=R_W} = -\sigma_0(\vartheta)/\varepsilon_0$$

$$-\sum_\ell (\ell+1)\frac{A_\ell R_W^{2\ell+1}}{R_W^{\ell+2}} P_\ell(\cos\vartheta) - \sum_\ell \ell A_\ell R_W^{\ell-1} P_\ell(\cos\vartheta) = -\frac{\sigma_0(\vartheta)}{\varepsilon_0}$$

$$\sum_{\ell=\phi}^{+\infty} (2\ell+1) A_\ell R_W^{\ell-1} P_\ell(\cos\vartheta) = \sigma_0(\vartheta)/\varepsilon_0$$

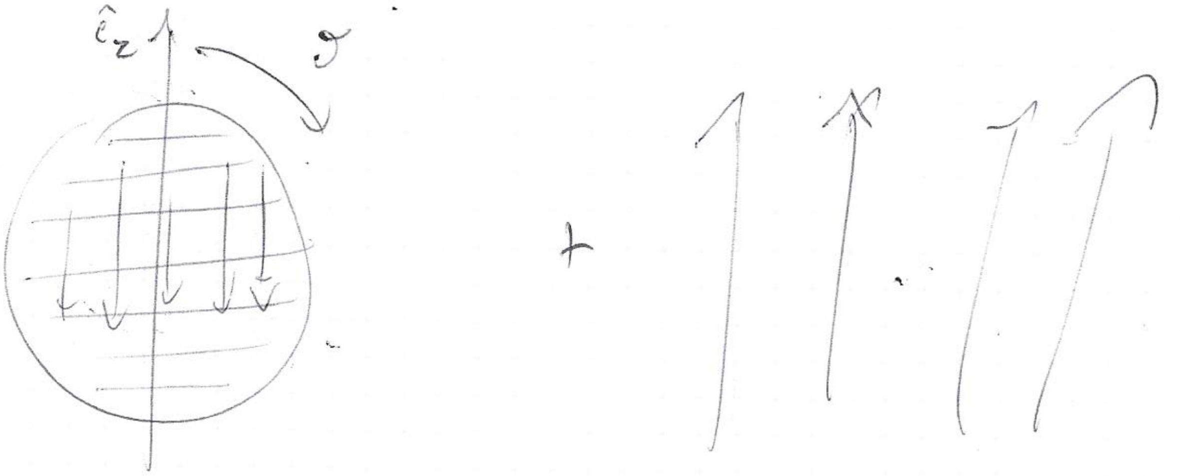
$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$

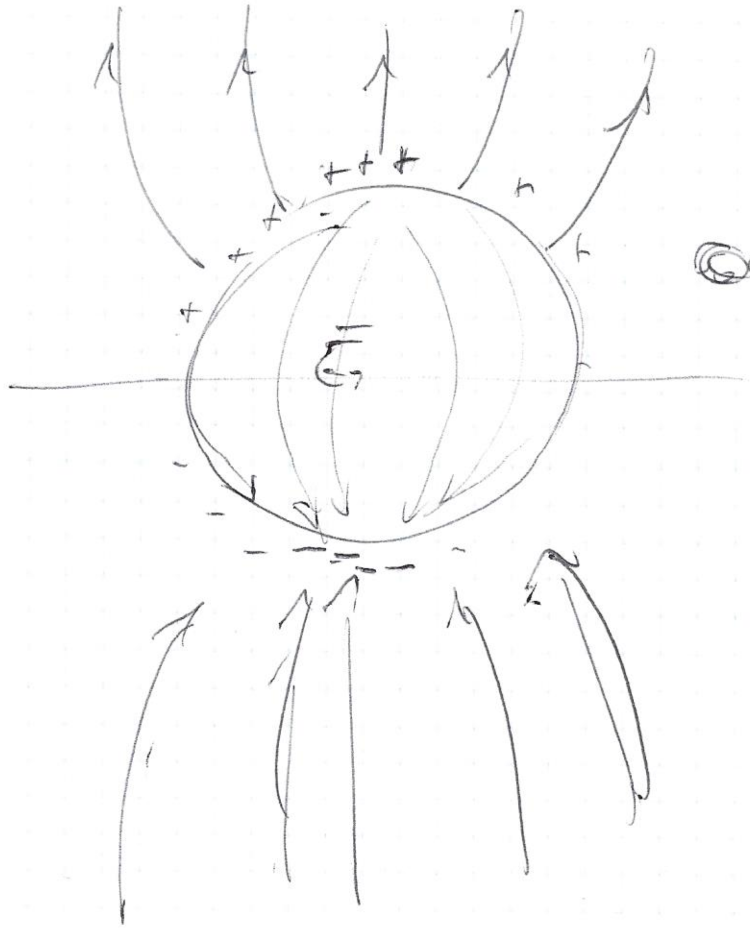
$$\sigma_0(\vartheta) = K\cos\vartheta = K P_1(\cos\vartheta)$$

$$A_\ell = \phi \quad \forall \ell \neq 1$$

$$\ell = 1 \quad 3A_1 \cancel{R_W^0} \underbrace{P_1(\cos\vartheta)}_{\cos\vartheta} = \frac{K}{\varepsilon_0}\underbrace{P_1(\cos\vartheta)}_{\cos\vartheta} \Rightarrow \boxed{A_1 = \frac{K}{3\varepsilon_0}}$$

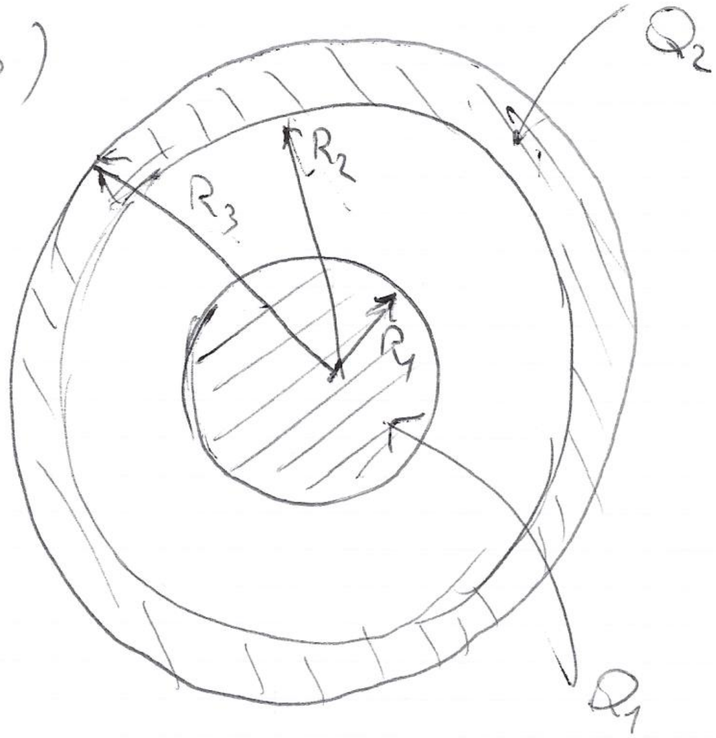
$$\begin{cases} V^{I}(r,\vartheta) = \dfrac{\kappa}{3\varepsilon_0}\,\overbrace{r\cos\vartheta}^{z} & \text{int.} \\[2ex] V^{II}(r,\vartheta) = \dfrac{\kappa R_W^3}{3\varepsilon_0}\,\dfrac{1}{r^2}\cos\vartheta & \text{ext.} \end{cases}$$





$\sigma \sim \cos\vartheta$

(5-3)



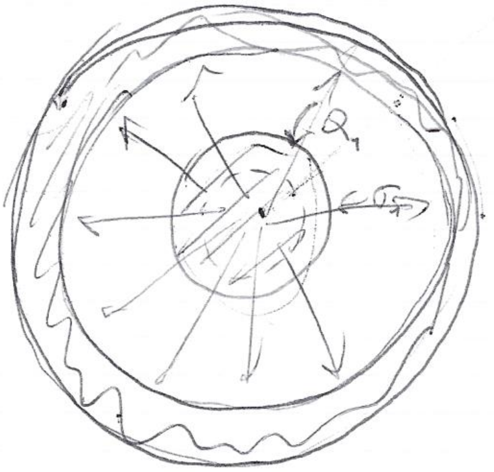
elettrostatica

$\bar{E}_0$ entro $\forall$ conduttore

$V_0(R_1)$, $V_0(R_3)$

$V_0(r=\infty)=\phi$

$R_1 = 10$ mm	$Q_1 = 10$ nC
$R_2 = 30$ ~~km~~	$Q_2 = 50$ nC
$R_3 = 40$ mm	



Q_1 su sup. R_1 $\quad \sigma_1 = \dfrac{Q_1}{4\pi R_1^2} = \dfrac{Q_1}{sup_1}$

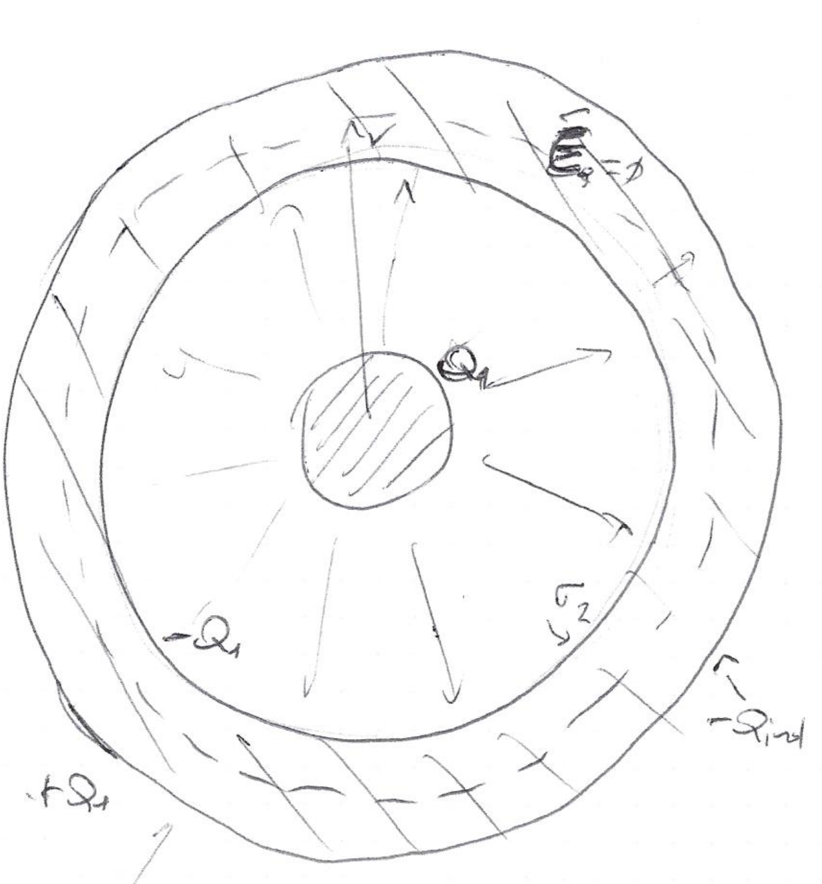
$\Phi(\bar{E}_0)\big|_{r<R_1} = \phi$
\+ simm. ~~sfer~~ sferica

$\bar{E}_0(r) = \phi$

induzione elettrostatica

induzione completa = tutte le linee di $\bar{E}$ vanno $1 \to 2$

Qindotta $= -Q$inducente



Gauss su sup. con r

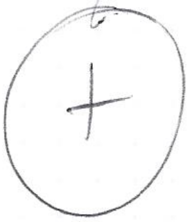
$R_1 < r < R_2$

$\Phi(\vec{E}_0) = 0$

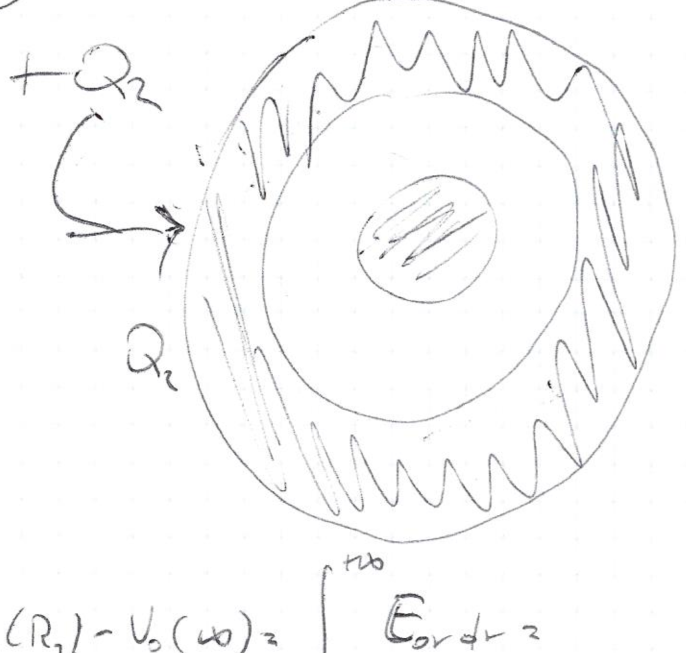
$\Rightarrow Q_{tot}$ racchiusa $= 0$

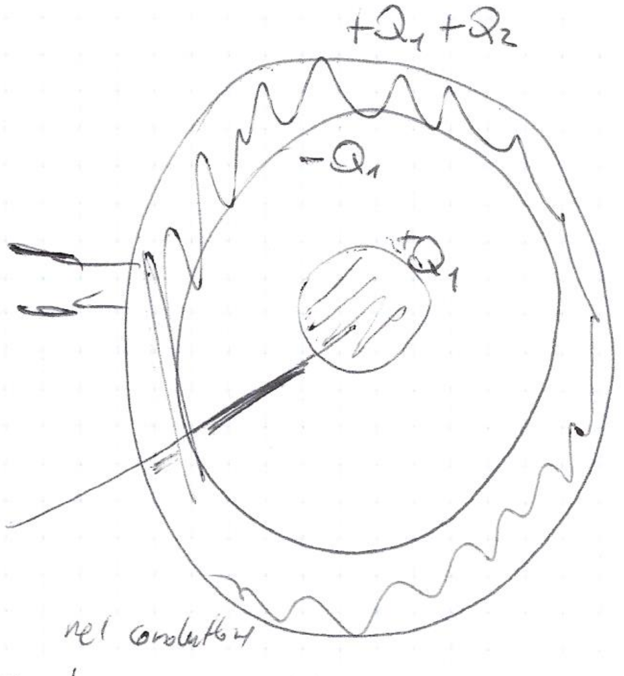
Q_1 (inducente) $+ Q_{ind} = 0$

$$\sigma_2 = Q_{ind}/4\pi R_2^2 = -\frac{Q_1}{4\pi R_2^2} = -8.84 \cdot 10^{-7}\ C/m^2$$



$$Q_{ind}^{(S2)} + (-Q_{ind})^{(S3)} = 0$$

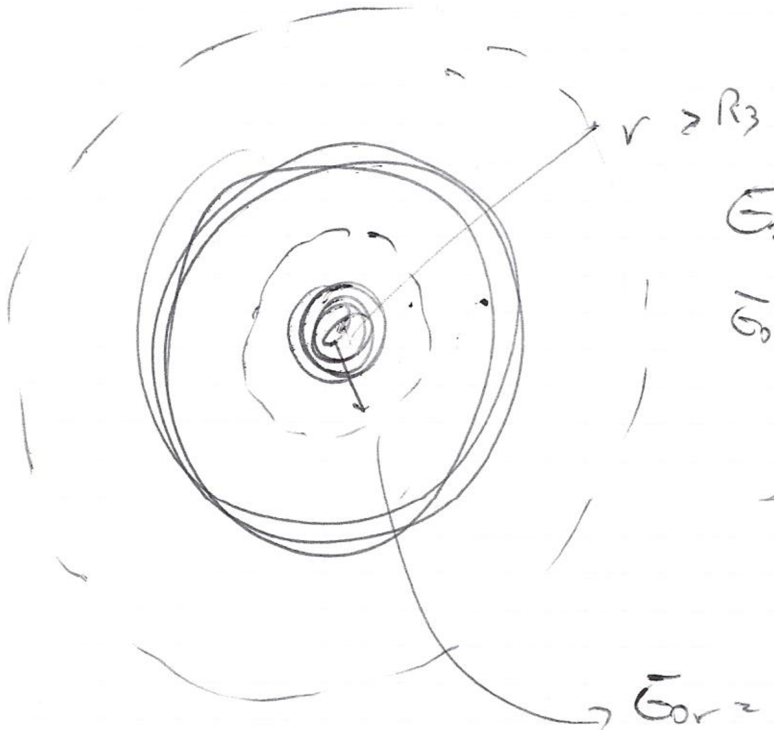




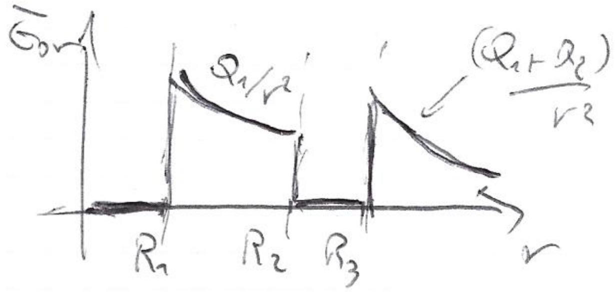
$$V_0(R_1) - V_0(\infty) = \int_{R_1}^{+\infty} E_0\,dr =$$

nel conduttore

$$= \int_{R_1}^{R_2} E_0\,dr + \cancel{\int_{R_2}^{R_3} E_0\,dr} + \int_{R_3}^{+\infty} E_0\,dr$$



$$E_{0r} = \frac{1}{4\pi\varepsilon_0}\frac{(Q_1+Q_2)}{r^2}$$



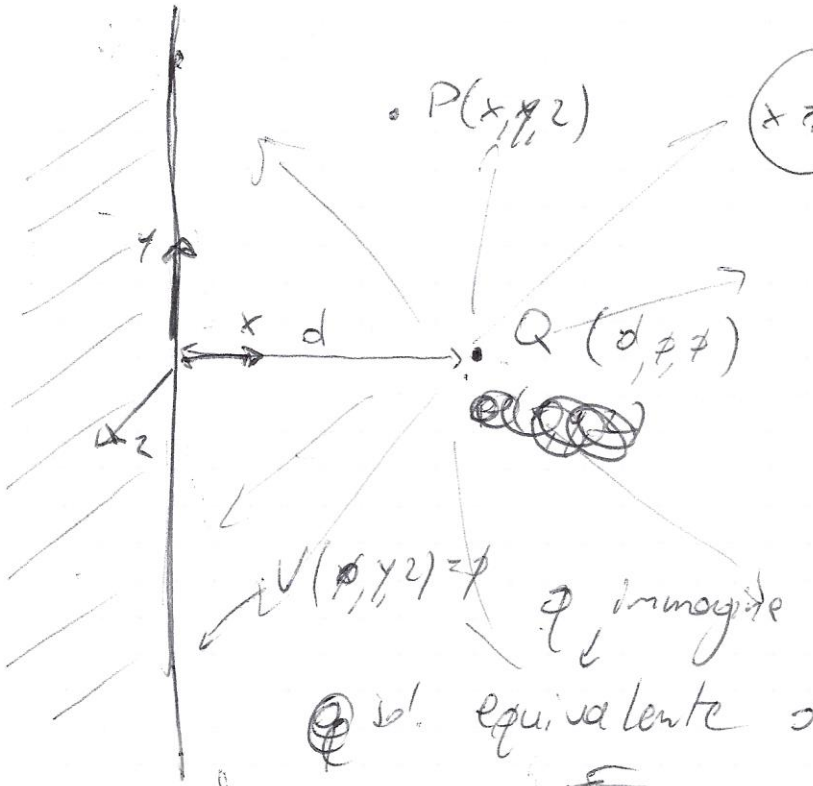
$$\rightarrow E_{0r} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}$$

$$V_0(R_1) = \frac{Q_1}{4\pi\varepsilon_0}\int_{R_1}^{R_2}\frac{1}{r^2}\,dr + \frac{Q_1+Q_2}{4\pi\varepsilon_0}\int_{R_3}^{+\infty}\frac{1}{r^2}\,dr =$$

$$= \frac{Q_1}{4\pi\varepsilon_0}\frac{R_2-R_1}{R_1R_2} + \frac{Q_1+Q_2}{4\pi\varepsilon_0}\frac{1}{R_3} = 22.47\ kV$$

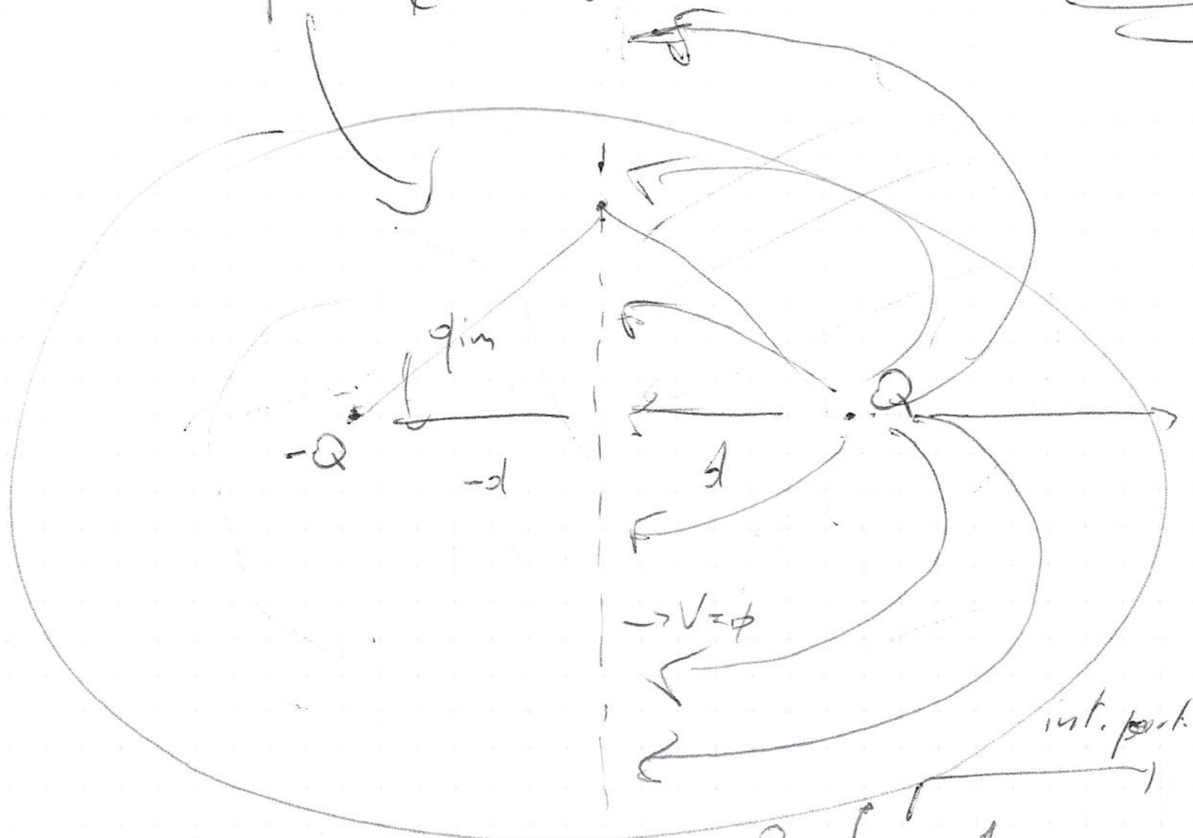
$$V_0(R_3) = \int_{R_3}^{+\infty} E_{0r}\,dr = \frac{Q_1+Q_2}{4\pi\varepsilon_0}\frac{1}{R_3} = 13.48\ kV$$

$$V_0(R_2) = V_0(R_3)$$



$V = \cancel{\frac{1}{4\pi\varepsilon} \frac{q}{|r - r_0|}}$ insufficiente

q sol. equivalente dove? Nel dominio



σ_{ind} ∝ E_{on}

$$V(x,y,z) = V_Q + V_{Qim} = \frac{Q}{4\pi\varepsilon_0}\left\{ \overbrace{\frac{1}{[(x-d)^2+y^2+z^2]^{\frac{1}{2}}}}^{\text{int. part.}} - \overbrace{\frac{1}{[(x+d)^2+y^2+z^2]^{\frac{1}{2}}}}^{\text{sol. omo.}} \right\}$$

→ $Q(d,0,0)$ → $-Q(-d,0,0)$

$\bar{r}_0 = (d,0,0)$ $V(0,y,z) = 0$ ok

$\nabla^2 V = 0$ omogenea ass.

$$\begin{cases} \nabla^2 V = -\frac{\rho}{\varepsilon_0} \to Q\delta(\bar{r}-\bar{r}_0) \\ V(0,y,z) = 0 \text{ b.c.} + V(r \to \infty, x>0) = 0 \end{cases}$$