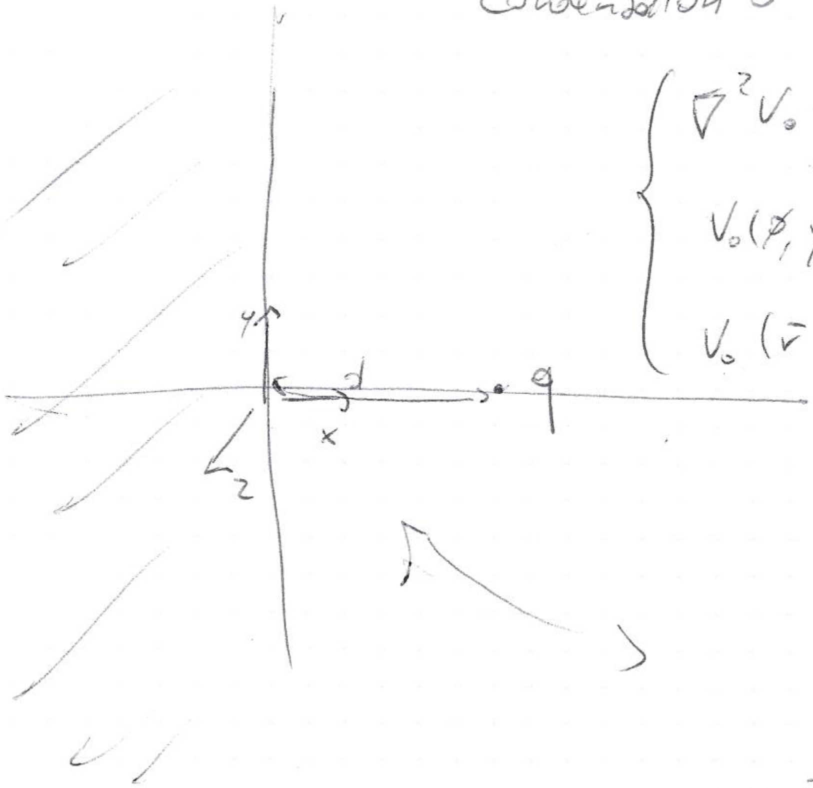
# ES. #9 Carica immagine; Condensatori

18/XI/2020

(1)




$$\begin{cases} \nabla^2 V_0 = -\rho/\varepsilon_0 \\ V_0(0, y, z) = 0 \\ V_0(\bar{r} \to +\infty,\ x \geq 0) = 0 \end{cases}$$

b.c. (last two conditions)

$x \geq 0$

$q' = -q \quad \xleftrightarrow{d} \quad | \quad \xleftrightarrow{d} \quad q$

$$V(x,y,z) = V_q + V_{q'} = \frac{q}{4\pi\varepsilon_0}\left\{ \underbrace{\frac{1}{[(x-d)^2+y^2+z^2]^{1/2}}}_{\text{sol. particolare di Poisson}} - \underbrace{\frac{1}{[(x+d)^2+y^2+z^2]^{1/2}}}_{\text{sol. di Laplace (omo-associata)}} \right\}$$

domain

$q$

$\nabla^2 V = -\rho/\varepsilon_0$

$+$

$q' = -q$

domain

$\nabla^2 V_0 = 0$

$E_n = \sigma/\varepsilon_0 \qquad E_n = E_x\big|_{x=0}$

$$E_{0x} = -\frac{\partial V}{\partial x} = -\frac{q}{4\pi\varepsilon_0}\left\{-\frac{1}{2}\frac{2(x-d)}{[(x-d)^2+y^2+z^2]^{3/2}} + \frac{1}{2}\frac{2(x+d)}{[(x+d)^2+y^2+z^2]^{3/2}}\right\}$$

in $x=0$

$$E_{0x}(0,y,z) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\cancel{x}-d}{[d^2+y^2+z^2]^{3/2}} - \frac{\cancel{x}+d}{[d^2+y^2+z^2]^{3/2}}\right\}$$

$$E_{0x}(0,y,z) = -\frac{q}{2\pi\varepsilon_0}\frac{d}{[d^2+y^2+z^2]^{3/2}} = -\frac{qd}{2\pi\varepsilon_0[d^2+r^2]^{3/2}}$$

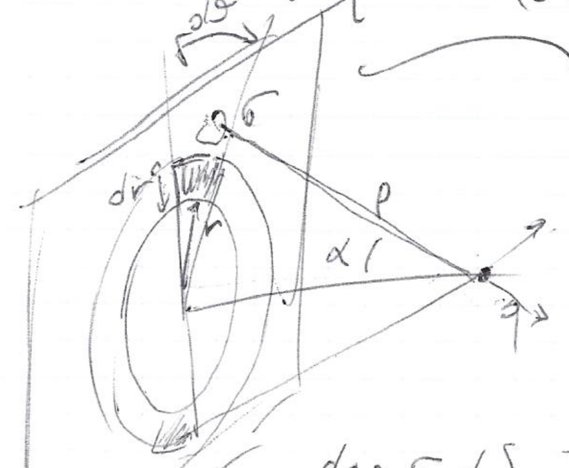
$$r \doteq \sqrt{y^2+z^2}$$

$$\sigma(0,\overbrace{y,z}^{r}) = \varepsilon_0 E_{0x}(0,y,z) = -\frac{q}{2\pi}\frac{d}{[d^2+r^2]^{3/2}}$$

q indotta totale

$$q_{ind} = \int_0^{+\infty}\sigma(r)\,2\pi r\,dr = -qd\int_0^{+\infty}\frac{r}{(d^2+r^2)^{3/2}}\,dr =$$

$$= -qd\underbrace{\left[\cancel{0} - \frac{1}{(d^2+r^2)^{1/2}}\right]_0^{+\infty}}_{\frac{1}{d}} = -q \quad (= q \text{ immagine})$$




$E_0$ della $\sigma$

$$d\vec F_c = q\,d\vec E_0$$

$$d\vec E_0 = \frac{1}{4\pi\varepsilon_0}\frac{\overbrace{\sigma(r)\,r\,dr\,d\theta}^{dq}}{\rho^2}\hat e_\rho$$

$$dq = \sigma\,dS$$

$$dE_{0x} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(r)\,2\pi r\,dr}{\rho^2}\cos\alpha$$

$$\cos\alpha = \frac{d}{\rho}$$

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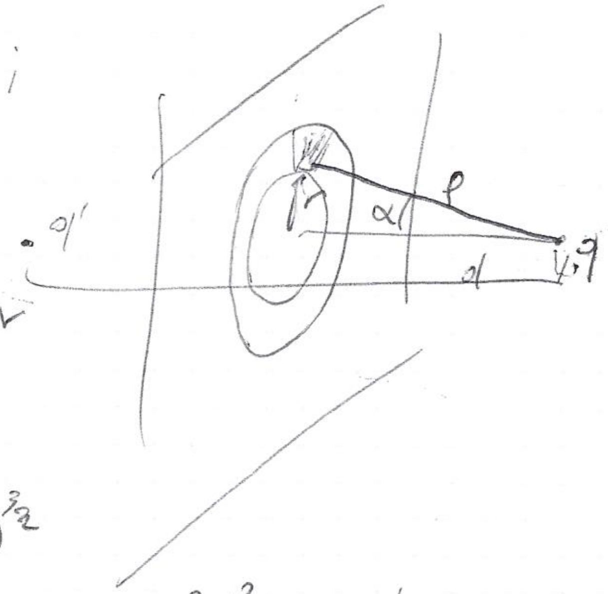
$$\cos\alpha = d/\rho \;;$$

$$\rho = (d^2+r^2)$$




$$dE_{0x} = \frac{1}{2\varepsilon_0}\,\frac{\sigma d}{(d^2+r^2)^{3/2}}\, r\,dr$$

$$\sigma(r) = -\frac{q}{2\pi}\,\frac{d}{(d^2+r^2)^{3/2}}$$

$$dF_x(r) = q\,dE_{0x} = -\frac{q^2 d^2}{4\pi\varepsilon_0}\,\frac{r\,dr}{(d^2+r^2)^3}$$

$$F_x = \int dF_x = -\frac{q^2 d^2}{4\pi\varepsilon_0}\int_0^{+\infty}\frac{r}{(d^2+r^2)^3}\,dr = -\frac{q^2 d^2}{4\pi\varepsilon_0}\left[-\frac{1}{4}\,\frac{1}{(d^2+r^2)^2}\right]_0^{+\infty}$$

$$F_x = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{4d^2} = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{D^2}$$

$$D = 2d$$

$$F_x = q_{reale}\cdot E_{0x}(q_{im}) = -\frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{D^2}$$

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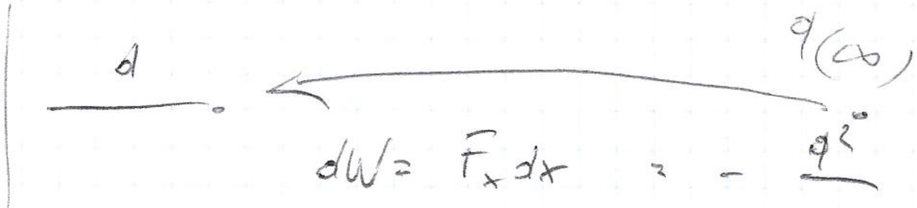
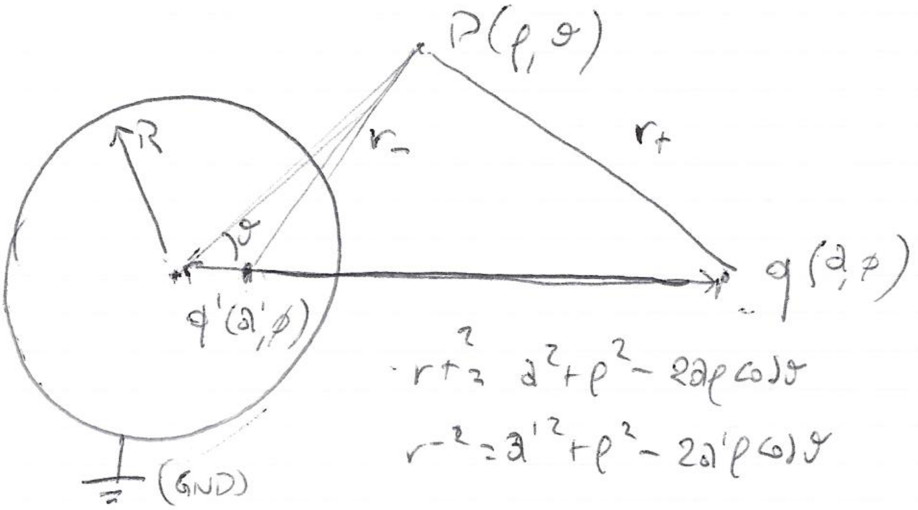



$$dW = F_x\,dx = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{(2x)^2}\,dx$$

$$W = \int_{+\infty}^{d} F_x\,dx = -\frac{q^2}{16\pi\varepsilon_0}\left[-\frac{1}{x}\right]_d^{+\infty} \Rightarrow$$

$$W = \frac{1}{16\pi\varepsilon_0}\,\frac{q^2}{d} = \frac{1}{8\pi\varepsilon_0}\,\frac{q^2}{2d}$$

En. di interazione $q \leftrightarrow -q$ $\quad U = -\dfrac{q^2}{4\pi\varepsilon_0(2d)} = 2W$ NO




b.c $\begin{cases} V_0(\infty) = \phi \\ V_0(R, \theta) = \phi \end{cases}$

$$r^{+2} = a^2 + \rho^2 - 2a\rho\cos\theta$$
$$r^{-2} = a'^2 + \rho^2 - 2a'\rho\cos\theta$$

$$V_0(\rho, \theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r^+} + \frac{q'}{r^-}\right] = \cancel{\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(a^2+\rho^2-2a\rho\cos\theta)^{\frac{1}{2}}} + \frac{q'}{(a'^2+\rho^2-2a'\rho\cos\theta)}\right]}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left[a^2+\rho^2-2a\rho\cos\theta\right]^{\frac{1}{2}}} + \frac{q'}{\left[a'^2+\rho^2-2a'\rho\cos\theta\right]^{\frac{1}{2}}}\right]$$

$V(R, \theta) = \phi$ = $f(\theta)$ costante

$$\frac{q'}{q} = -\left[\frac{a'^2 + R^2 - 2a'R\cos\theta}{a^2 + R^2 - 2aR\cos\theta}\right]^{\frac{1}{2}} =$$

$$= \left(\frac{a'^2+R^2}{a^2+R^2}\right)^{\frac{1}{2}} \underbrace{\left[\frac{\left(1 - \left(\frac{2a'R\cos\theta}{a'^2+R^2}\right)\right)}{\left(1 - \left(\frac{2aR\cos\theta}{a^2+R^2}\right)\right)}\right]^{\frac{1}{2}}}_{f(\theta)}$$

$$\frac{2a'R}{a'^2+R^2} = \frac{2aR}{a^2+R^2}$$

$$\ldots \quad \underbrace{(a'a - R^2)}(a - a') = \phi$$

banale ~~$a' = a$~~ $\quad \boxed{a' = R^2/a}$

$$\frac{q'}{q} = -\left[\frac{a'^2+R^2}{a^2+R^2}\right]^{\frac{1}{2}} = -\left[\frac{\frac{R^4}{a^2}+R^2}{a^2+R^2}\right]^{\frac{1}{2}} = -\frac{R}{a}\left(\cancel{\frac{R^2+a^2}{a^2+R^2}}\right)^{\frac{1}{2}}$$

$a' = R^2/a$

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$$\boxed{q' = -\frac{R}{a}\,q} \checkmark$$

$$\boxed{a' = R^2/a} \checkmark$$

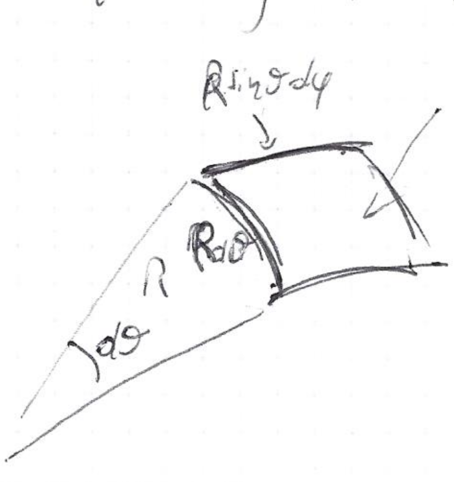
$$\boxed{\sigma_{ind}(\theta)} = \varepsilon_0 E_{0\rho}(\rho=R,\theta) = -\varepsilon_0 \left.\frac{\partial V_0}{\partial \rho}\right|_{\rho=R} =$$

$$= -\varepsilon_0 \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial\rho}\left[\frac{1}{(a^2+\rho^2-2a\rho\cos\theta)^{\frac{1}{2}}} - \frac{1}{(R^2+(a\rho/R)^2-2a\rho\cos\theta)^{\frac{1}{2}}}\right]_R$$

$$= \frac{q}{4\pi}\left[\frac{1}{2}\frac{2\rho-2a\cos\theta}{(a^2+\rho^2-2a\rho\cos\theta)^{3/2}} - \frac{1}{2}\frac{2a^2\rho/R^2-2a\cos\theta}{(R^2+(a\rho/R)^2-2a\rho\cos\theta)^{3/2}}\right]_{\rho=R} =$$

$$= \frac{q}{4\pi}\left[\frac{R-a^2/R}{(a^2+R^2-2aR\cos\theta)^{3/2}}\right] = \boxed{-\frac{q}{4\pi}\frac{a^2-R^2}{R(a^2+R^2-2aR\cos\theta)^{3/2}}}$$

$$q_{ind} = \int \sigma_{ind}\, dS_{sph}$$




$$dS = R^2 \sin\theta\, d\theta\, d\varphi$$

$$q_{ind} = \int_0^{2\pi}\int_0^{\pi} \sigma_{ind}(\theta)\, dS = \int_0^{\pi} \sigma_{ind}(\theta)\, 2\pi R^2 \sin\theta\, d\theta$$

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$$q_{ind} = -\frac{q}{4\pi}\,\frac{a^2-R^2}{R}\,2\pi R^2 \int_0^{\pi} \frac{\sin\vartheta}{(a^2+R^2-2aR\cos\vartheta)^{3/2}}\,d\vartheta =$$

$$\frac{d}{d\vartheta}\left[(a^2+R^2-2aR\cos\vartheta)^{-\frac{1}{2}}\right] = -aR\,\frac{\sin}{(\dots)^{3/2}}$$

$$= \frac{q}{2}\,R(a^2-R^2)\left[\frac{1}{aR}\,(a^2+R^2-2aR\cos\vartheta)^{-\frac{1}{2}}\right]_0^{\pi} =$$

$$= \frac{q}{2}\,\frac{a^2-R^2}{a}\left(\underbrace{\frac{1}{(a^2+R^2+2aR)^{\frac{1}{2}}}}_{((a+R)^2)^{\frac{1}{2}}} - \underbrace{\frac{1}{(a^2+R^2-2aR)^{\frac{1}{2}}}}_{((a-R)^2)^{\frac{1}{2}}}\right] =$$

$$= \frac{q}{2a}\left[(a-R)-(a+R)\right] = -\frac{R}{a}\,q = q'_{im}$$

F idu q reale

$$F = \frac{qq'}{4\pi\varepsilon_0 d} = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{R}{a}\,\frac{a^2}{(a^2-R^2)^2}$$

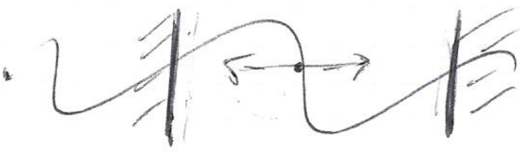
$\frac{1}{d^2}$

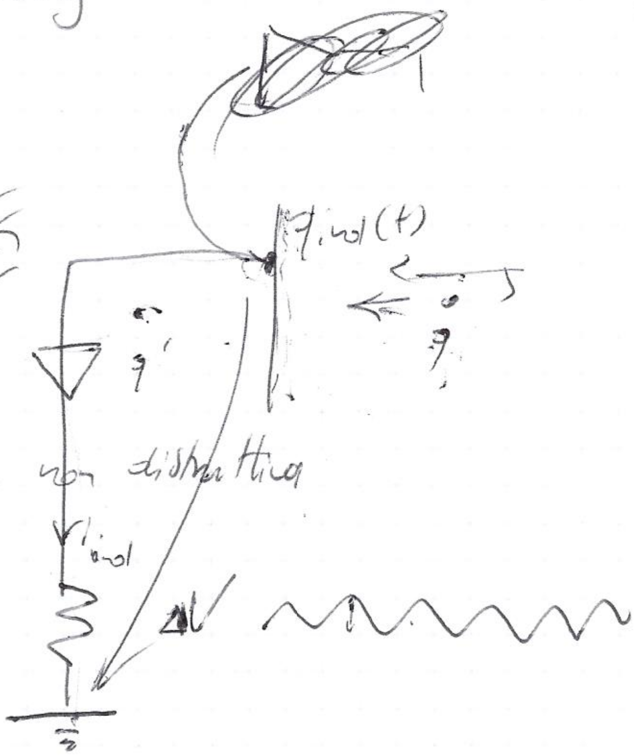
$$d = a - a' = \frac{a^2-R^2}{a}$$

$$\Rightarrow\quad F = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{aR}{(a^2-R^2)^2}$$

Trappola di Penning

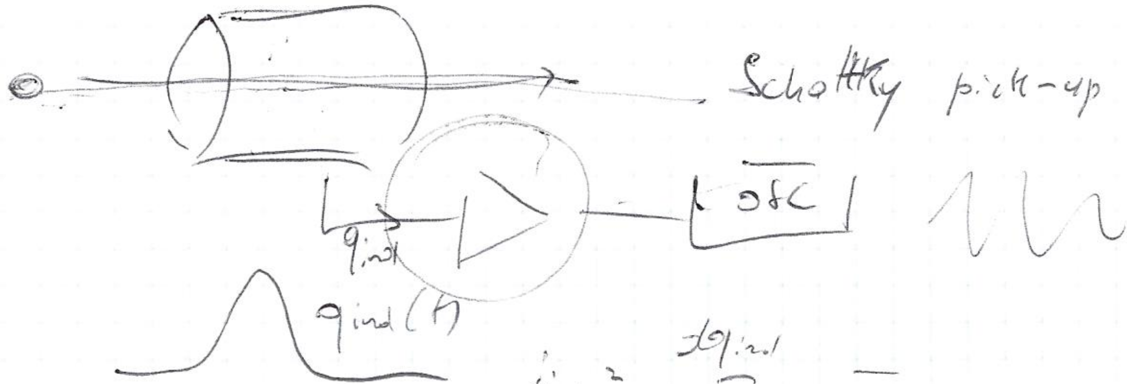
$V_0 + \vec{B}$






diagnostica non distruttiva

Schottky pick-up




$\rightarrow i_{ind} = \dfrac{dq_{ind}}{dt}$

# Condensatori elettrostatici

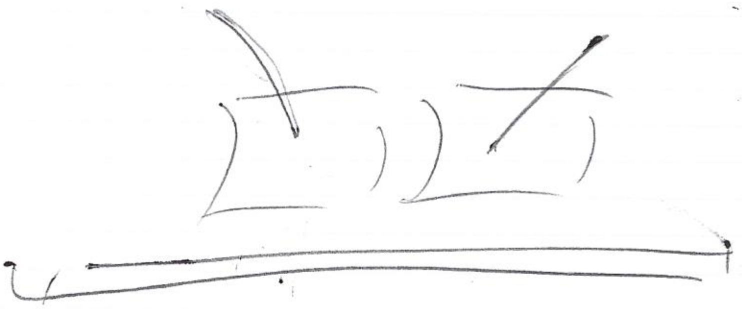
capacitori (capacitor)

Cond.: sistema di due conduttori in regime di induzione completa

Armature (intercapedine)

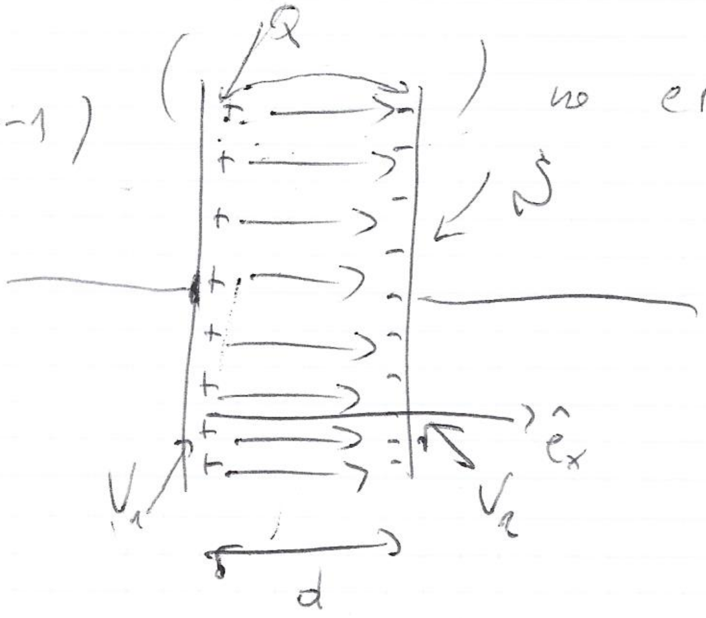
Capacità: $C = \frac{Q}{\Delta V}$ $[C] = \left[\frac{\text{Coulomb}}{\text{Volt}}\right] = [F]$

Farad

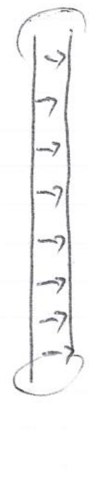


capacità parassite

(7-1)




no effetti di bordo (fringe field)



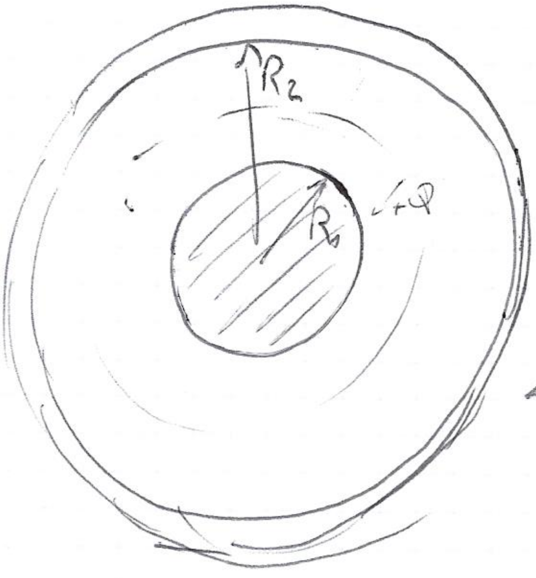
$$E_0 = \frac{\sigma}{\varepsilon_0}\hat{e}_x = \frac{Q}{\varepsilon_0 S}\hat{e}_x$$

$$\Delta V = V_1 - V_2 = \int_1^2 \bar{E}\cdot d\bar{l} = \int_0^d E_{0x}\,dx = \frac{Q}{\varepsilon_0 S}\int_0^d dx = \frac{Qd}{\varepsilon_0 S}$$

$$\boxed{C = \frac{Q}{\Delta V} = \frac{\varepsilon_0 S}{d}}$$ $S = 10 \times 10$ mm $d = 1$ mm $C = \underline{0.89\ pF}$

(7-2) Cond. sferico






$$\vec{E}_0(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{e}_r$$

$$Q = 4\pi R_1^2 \sigma$$

$$\Delta V = V_1 - V_2 = \int_{R_1}^{R_2} E_{0r}\, dr =$$

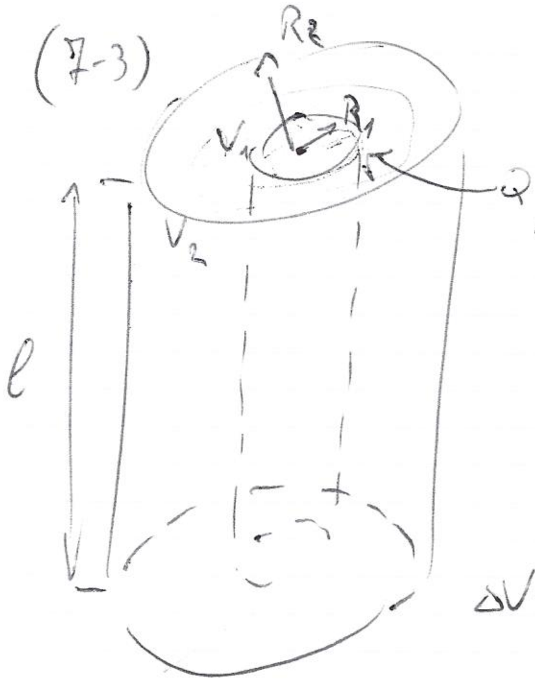
$$= \frac{Q}{4\pi\varepsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

$$C = Q/\Delta V = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}$$

$\delta \doteq R_2 - R_1$ $\delta \ll R_1, R_2$ $R \simeq R_1 = R_2$

$$C \simeq \frac{4\pi\varepsilon_0 R^2}{\delta} = \frac{\varepsilon_0 S}{\delta}$$

(7-3)




$\ell \gg R_1, R_2$

$\lambda = Q/\ell$

$$\vec{E}_0(r) = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{e}_r \qquad r \in (R_1, R_2)$$

$$\Delta V = V_1 - V_2 = \int_{R_1}^{R_2} E_{0r}\,dr = \frac{\overset{\rightarrow Q/\ell}{\lambda}}{2\pi\varepsilon_0}\log(R_2/R_1)$$

$$C = \frac{Q}{\Delta V} = \frac{\lambda\ell}{\Delta V} = \frac{2\pi\varepsilon_0\ell}{\log(R_2/R_1)}$$

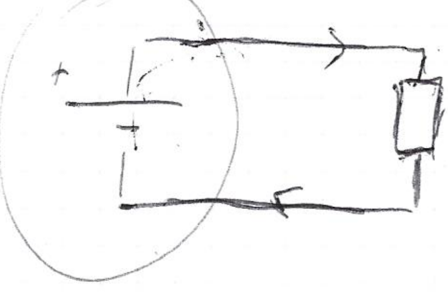
$\ell = 10\,mm$, $R_1 = 2.8\,mm$, $R_2 = 3\,mm$ $\quad C = 8.1\,pF$

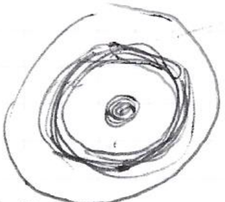
$$\delta \doteq R_2 - R_1 \ll R \simeq R_1, R_2$$

$$\log\left(\frac{R_2}{R_1}\right) = \log\left(1 - 1 + \frac{R_2}{R_1}\right) = \log\left(1 + \frac{R_2 - R_1}{R_1}\right) = \log\left(1 + \frac{\delta}{R}\right) \simeq \frac{\delta}{R}$$

$$\Rightarrow C \simeq \frac{2\pi\varepsilon_0\ell R}{\delta} = \frac{\varepsilon_0 S}{\delta}$$

$2\pi R\ell = S_{cyl}$





 = cond.

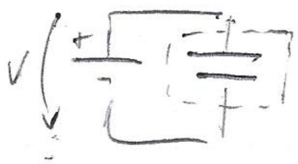
 = guaina isolante

cavo 
coassiale

11

$\overline{E}$n. elettrostatica contenuto in condensatore

$\phi \to dQ$ portato su armatura




$$dW = V\,dQ = \frac{1}{C} Q\,dQ$$

ddp tra poli generatore

$$C = Q/V$$

$\phi \to Q$
$$W = \int_0^Q \frac{1}{C} Q'\,dQ' = \frac{1}{2}\frac{Q^2}{C} = U_e$$

$$U_e = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} QV = \frac{1}{2} CV^2$$

$$= \frac{1}{2}\frac{d}{\varepsilon_0 S} Q^2 = \frac{1}{2}\varepsilon_0 \underbrace{S d}_{vol} \underbrace{\left(\frac{Q}{\varepsilon_0 S}\right)^2}_{E_0^2} = \frac{1}{2}\varepsilon_0 E_0^2\, vol = u_e \cdot vol$$

$$E^2 = \vec{E}\cdot\vec{E}$$

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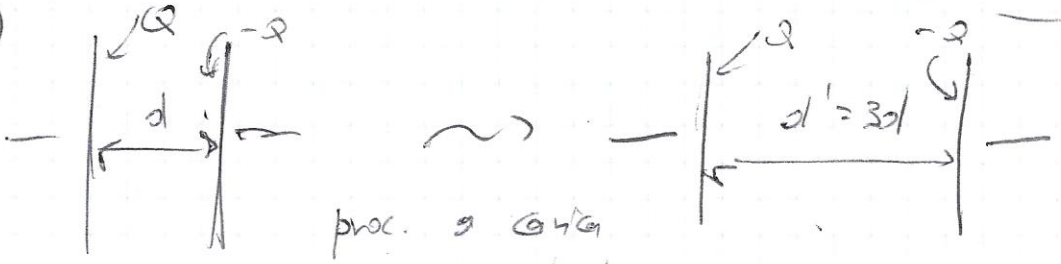
(7-4)




proc. a carica costante

$$U_e = \frac{1}{2}\frac{Q^2}{C} \qquad U_e' = \frac{1}{2}\frac{Q^2}{C'}$$

$$\frac{U_e'}{U_e} = \frac{C}{C'} = \frac{\varepsilon_0 S}{d}\cdot\frac{d'}{\varepsilon_0 S} = \frac{d'}{d} = 3 \qquad \left(\frac{V'}{V} = \frac{Q}{C'}\cdot\frac{C}{Q} = \frac{d'}{d} = 3\right)$$