Introduction to Continuum Physics - General notions (Water flea or hippo?)

What is a continuous medium? It is a region of matter, a material medium where, once we establish the maximum distance d that we can consider as infinitesimal, if we consider a volume of diameter d the number of microscopic constituent units or particles is large enough to be statistically significant. [Diameter is the maximum distance between any pair of points within the volume under consideration.]

Inevitably, this is not an absolute definition; on the contrary, a medium is to be considered as a continuum (or not) depending on the comparison between size scales. Here are some examples (of increasingly large scale).

= In solids and liquids, interatomic / intermolecular distances are in the order of 1-10 Å, i.e. 0.1-1 nm; in a cubic volume of 1 μm side we get > 10^9 constituent particles. But is 1 μm an infinitesimal scale?

* For a nanotechnologist who investigates matter at submicrometric scales, that is a no!
* When investigating a macroscopic fluid flow, e.g. in a laboratory experiment or a natural flow, size and obstacles in the flow can range from one meter (e.g., a bridge pillar) to granular suspensions (10-100 μm) at most, hence 1 μm is indeed small (infinitesimal) and we can treat the medium as continuous.

= In a gas like air, a mole (i.e. a number of particles equal to Avogadro's constant, $\sim 10^{23}$) in normal conditions (1 atm, 0°C) fills a volume of ~22 L, i.e. a cube with ~0.28 m side, i.e. $\sim 10^5$ μm; a cube with a 1 μm side hence contains $\sim 10^{23}/10^{15} = 10^8$ molecules, that is still a statistically high number ⟹ continuum approximation still holds when we study a gas at the laboratory or even geophysical scale.

Yet, when considering geophysical phenomena, we must take into account the rarefaction of the atmosphere with increasing altitude; in its upper part, the Earth's atmosphere (say beyond 100 Km) drops in pressure and density by several orders of magnitude and statistic becomes insufficient, to the point where a fluid (continuum) description is inadequate and must be replaced by kinetic theory.

= Scales become really extreme in an astrophysical context. The average density of the

interstellar medium is $\sim 10^6$ particles/m^3, so it is <u>not</u> a continuous medium at the scale of a human-made space probe.
At the galactic scale, we can adopt the light year ($\sim 10^{16}$ m) as the reference length and something like 10^{-9} light years $= 10^7$ m is an infinitesimal length. In a volume of diameter 10^7 m there is a statistically significant number of particles (among which, even in a small fraction, we could find charged particles) and a fluid description is fine again.
A note: we do the same in classical electrodynamics when we define charge and current densities (λ, σ, ρ; $\bar{K}, \bar{J}$) and the macroscopic vector fields called polarization and magnetization densities ($\bar{P}, \bar{M}$): Once again, we approximate matter (which is made of atoms and hence inherently discrete) as continuous.

A sizeable fraction of this course is devoted to continuous media called <u>fluids</u>, i.e. continuous media that cannot remain in equilibrium (and are thus set in motion) when subjected to shear stresses (we will come back to this in more detail). They roughly include liquids and gases, while elastic solids, albeit continuous media themselves, undergo a different definition and description; yet materials exist whose behaviour challenges such a sharp distinction.
The state of a continuum is not completely identified with the mere dynamics: A full <u>thermodynamic</u> study of the system is required. We will devote part of the course to the description of heat transfer, dealing at large with <u>conduction</u> (a heat exchange process relevant mostly to solids – and we will see some geophysically-relevant examples) and touching some aspects of <u>convection</u>, i.e. the main heat transfer mechanism in fluids.
Starting with fluids allows (or forces) us to deal with some delicate aspects of our physical investigation. How can we study a fluid? There are two points of view.
<u>Eulerian</u> approach: We observe the temporal variation of the quantities of interest in a set point $\bar{x}$. If we have a flow, in the region around $\bar{x}$ the fluid contained in this region is different at each time instant t.

Lagrangian approach: We follow a fluid particle along its evolution, along its motion. Hence the position $\bar{x}$ associated with the region of fluid around it is actually $\bar{x}(t)$.

Note: We define fluid particle a volume of continuum of infinitesimal linear size. It is basically a point particle with its mass.

The Lagrangian approach requires us to be careful when we use time derivatives, i.e. in any mathematical operation usually exploited to study the dynamics of an object. Given an extensive quantity $F(\bar{x},t)$, taking its time derivative in a Lagrangian approach, i.e. following the fluid particle found in $\bar{x}$ at time t, means taking into account the fact that F depends on t both explicitly and implicitly, through $\bar{x}(t)$, so $F = F(\bar{x}(t), t)$.

Hence we must deal with multivariate function composition, and the derivative of such composition, denoted D/Dt, is called MATERIAL or ADVECTIVE or SUBSTANTIAL DERIVATIVE.

The rule for function composition yields

$$\frac{D}{\partial t} F(\bar{x}(t),t) = \frac{\partial}{\partial t} F(\bar{x}(t),t) + \sum_{i=1}^{3} \underbrace{\frac{\partial F(\bar{x},t)}{\partial x_i}}_{(\text{grad } F)_i} \underbrace{\frac{dx_i(t)}{dt}}_{v_i(t)} = \frac{\partial F(\bar{x}(t),t)}{\partial t} + \bar{v}(\bar{x}(t),t) \cdot \text{grad} F(\bar{x}(t),t)$$

or, if we want to show it more explicitly,

we have $F(\bar{x}(t),t) = F(g(t))$

where $g(t) = (x_1(t), x_2(t), x_3(t), t) = (g_1, g_2, g_3, g_4)$

$$\Rightarrow \dot{g}(t) = \frac{d}{dt} g(t) = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, 1\right) = (v_1, v_2, v_3, 1)$$

$$\Rightarrow \frac{d}{dt} F(g(t)) = \frac{\partial F}{\partial g_1}\dot{g}_1 + \frac{\partial F}{\partial g_2}\dot{g}_2 + \frac{\partial F}{\partial g_3}\dot{g}_3 + \frac{\partial F}{\partial g_4}\dot{g}_4 =$$

$$= \frac{\partial F}{\partial x_1} v_1 + \frac{\partial F}{\partial x_2} v_2 + \frac{\partial F}{\partial x_3} v_3 + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial t} + \bar{v} \cdot \text{grad} F(\bar{x},t)$$

This derivative taken along the fluid particle motion retain all properties of derivatives; for instance we can write down to relationship between differential and material derivative, i.e. we can express the variation DF of the quantity F occurring while following the particle motion; to first order,

$$DF(\bar{x}(t),t) = F(\bar{x}(t+dt), t+dt) - F(\bar{x}(t),t) = \frac{DF}{Dt}(\bar{x}(t),t)\, dt$$

As an example of material derivative let us write down the inertial acceleration. This one is defined (in its components)

indeed the differential of $\bar{v}$ since $t' = t + dt$ with dt infinitesimal

$$a_i = \lim_{t \to t'} \frac{1}{t'-t} \left[v_i(\underbrace{\bar{x}(t')}_{}, t') - v_i(\bar{x}(t), t) \right]$$

position $\bar{x}(t')$ = time evolution in t' of position $\bar{x}$ at time t

$$\Rightarrow a_i = \frac{D}{Dt} v_i(\bar{x}(t), t) = \frac{\partial v_i}{\partial t} + (\bar{v} \cdot \text{grad}) v_i$$ or in vector form

$$\bar{a}(\bar{x}(t), t) = \frac{D}{Dt} \bar{v}(\bar{x}(t), t) = \frac{\partial}{\partial t} \bar{v}(\bar{x}(t), t) + (\bar{v} \cdot \text{grad}) \bar{v}(\bar{x}(t), t)$$