Gravity waves

Gravity waves are wave phenomena where the restoring force counteracting the perturbation is indeed gravity. These appear mainly as surface phenomena (SURFACE WAVES), where the perturbation of the free surface of a fluid initiates a wave motion (which is damped at increasing depths), but INTERNAL WAVES due to density inhomogeneities also occur.

We shall give a treatment of gravity waves within the approximation of <u>ideal fluids</u>, for the special case of <u>incompressible potential flow</u>. The essential issue to address to deal with this phenomenon concerns the boundary conditions at the interface between two fluids (or at the free surface between a fluid and a lighter medium approximated as vacuum).

- General form of kinematic conditions – ideal fluids

When we consider two immiscible fluids, we call interface their separation surface at any time; the most general form expressing geometrically this surface will be implicit, i.e. $F(\bar{x},t)=\phi$. By its own definition, $\text{grad}F$ is orthogonal to surfaces $F=\text{constant}$, therefore we can define the unit normal vector of the interface at point $\bar{x}$, time t as

$$\hat{n}(\bar{x},t)=\left.\frac{\text{grad}F(\bar{x},t)}{|\text{grad}F(\bar{x},t)|}\right|_{F=\phi}$$

If the point $\bar{x}$ at time t moves to $\bar{x}+d\bar{x}$ at $t+dt$, the differential of F is

$$dF=F(\bar{x}+d\bar{x},t+dt)-F(\bar{x},t)=\text{grad}F\cdot d\bar{x}+\frac{\partial F}{\partial t}dt$$

and both are $=\phi$ since $\bar{x}$, $\bar{x}+d\bar{x}$ are points on the interface; calling the velocity of a point $\bar{x}$ on the interface $\bar{v}\doteq d\bar{x}/dt$, we get

$$\bar{v}\cdot\text{grad}F+\frac{\partial F}{\partial t}=\phi$$

and this equation defines the velocity of the interface, or more precisely the component of $\bar{v}$ that is orthogonal to the interface, as $v_n=\bar{v}\cdot\text{grad}F/|\text{grad}F|$.

Now let us consider two <u>ideal fluids</u> (1), (2). Since the interface is unique, that is to say the boundary surfaces of the two fluids must coincide and cannot separate, the v_n at the interface must be the same for (1) and (2). So we can write

$$\left\{\begin{array}{l} \left.\left(\bar{v}_1 \cdot \text{grad} F + \dfrac{\partial F}{\partial t}\right)\right|_{F=0} = 0 \\ \\ \left.\left(\bar{v}_2 \cdot \text{grad} F + \dfrac{\partial F}{\partial t}\right)\right|_{F=0} = 0 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} v_{1n} = - \left.\dfrac{1}{|\text{grad} F|} \dfrac{\partial F}{\partial t}\right|_{F=0} \\ \\ v_{2n} = - \left.\dfrac{1}{|\text{grad} F|} \dfrac{\partial F}{\partial t}\right|_{F=0} \end{array}\right.$$

- General form of dynamic conditions – ideal fluids, incompressible potential flow

When the form of the interface is known at all times, kinematic conditions are enough, because one obtains v_n and the interface boundary conditions (i.b.c.) are perfectly defined (e.g., for a potential flow we obtain Neumann conditions for the Laplace equation $\nabla^2\varphi = 0$).

Nevertheless, the interface is unknown in general, and additional conditions are required. These are called dynamic conditions.

Now let us consider two ideal fluids (1), (2). We have seen that when surface tension can be neglected, pressure must be continuous across the interface. This is indeed the additional condition we must ask for:

$$p_1 = p_2 \quad \text{at } F(\bar{x}, t) = 0$$

If both fluids are in a state of incompressible potential flow, we can write the generalized Bernoulli equation $\dfrac{\partial \varphi}{\partial t} + \dfrac{1}{2}|\text{grad}\,\varphi|^2 + \psi + u = f(t)$

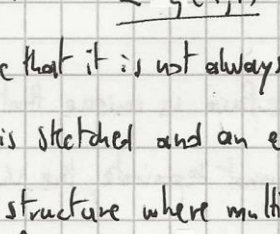
and using a suitable gauge such that $f(t) = 0$, having $\psi = p/\rho$ and $u = gz$ (vertical gravity), the dynamic condition $p_1 = p_2$ can be recast in the form

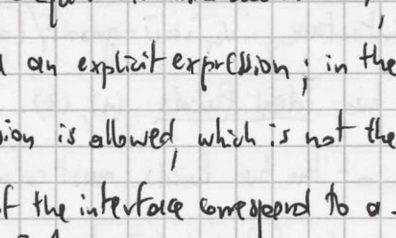
$$-p_1 = \rho_1 \frac{\partial \varphi_1}{\partial t} + \frac{1}{2}\rho_1 |\text{grad}\,\varphi_1|^2 + \rho_1 g z = \rho_2 \frac{\partial \varphi_2}{\partial t} + \frac{1}{2}\rho_2 |\text{grad}\,\varphi_2|^2 + \rho_2 g z = -p_2 \quad \text{for } F(\bar{x}, t) = 0$$

- Conditions for an explicit-form interface

The interface is expressed in an explicit form if the expression $F(\bar{x}, t) = 0$ can be rewritten in a form $\underline{z = \zeta(\bar{x}, t)}$ (where the equilibrium interface is $z = 0$, given gravity $\bar{g} = -g\hat{e}_z$).

Notice that it is not always possible find an explicit expression; in the left figure, a small-amplitude wave is sketched and an explicit expression is allowed, which is not the case at wave break (a non-linear structure where multiple points of the interface correspond to a single x) on the right.





So $F(\bar{x},t) = z - \zeta(x,y,t) = \phi$ and the kinematic conditions can be rewritten starting from the expression of grad F:

$$\operatorname{grad} F = \frac{\partial F}{\partial x}\hat{e}_x + \frac{\partial F}{\partial y}\hat{e}_y + \frac{\partial F}{\partial z}\hat{e}_z = -\frac{\partial \zeta}{\partial x}\hat{e}_x - \frac{\partial \zeta}{\partial y}\hat{e}_y + \hat{e}_z$$

From now on we shall assume a y-invariance (so that the interface and all quantities shall be independent of y : $v_{1,2_y} = \phi \Rightarrow z = \zeta(x,t)$ interface ; $\bar{v}_{1,2} = \bar{v}_{1,2}(x,z,t)$).

The kinematic conditions become

$$\begin{cases} \left(\dfrac{\partial F}{\partial t} + \bar{v}_1 \cdot \operatorname{grad} F\right)_{F=\phi} = \phi \\ \left(\dfrac{\partial F}{\partial t} + \bar{v}_2 \cdot \operatorname{grad} F\right)_{F=\phi} = \phi \end{cases} \rightarrow \boxed{\begin{cases} \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial \zeta}{\partial x} v_{1x}\Big|_{z=\zeta} - v_{1z}\Big|_{z=\zeta} = \phi \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial \zeta}{\partial x} v_{2x}\Big|_{z=\zeta} - v_{2z}\Big|_{z=\zeta} = \phi \end{cases}}$$

The dynamic condition is also recast as

$$\boxed{\left(\rho_1 \frac{\partial \varphi_1}{\partial t} + \frac{1}{2}\rho_1 |\operatorname{grad}\varphi_1|^2\right)_{z=\zeta} + \rho_1 g \zeta = \left(\rho_2 \frac{\partial \varphi_2}{\partial t} + \frac{1}{2}\rho_2 |\operatorname{grad}\varphi_2|^2\right)_{z=\zeta} + \rho_2 g \zeta}$$

Linearization of interface conditions – incompressible potential flow

The i.b.c. as stated so far are not linear, and since we want to find analytic solutions in the linear regime, i.b.c. must equally be reduced to linear ones. To this aim, we make a very straight assumption: All quantities of interest, as well as their spatial and temporal derivatives, experience deviations with respect to the values at mechanical equilibrium that are first-order infinitesimal variations.

To say this more explicitly, said $z=\phi$ the equilibrium height of the equilibrium interface,

$\Rightarrow \zeta$ = height with respect to equilibrium is a 1st-order infinitesimal quantity;

$p_0(\bar{x})$ = equilibrium pressure $\Rightarrow p'(\bar{x},t) \doteq p(\bar{x},t) - p_0(\bar{x})$ 1st-order infinitesimal quantity;

$\partial\zeta/\partial t$, $\partial\zeta/\partial x_j$, $\bar{v}$, $\partial_j v_i$: 1st-order infinitesimal quantity;

$\partial\varphi/\partial t$, $\partial\varphi/\partial x_j$ (for potential flows) 1st-order infinitesimal quantities.

Let us see the consequences of this hypothesis on the i.b.c. for incompressible potential flows.

Kinematic conditions

We start off by eliminating evident quadratic terms in the variables → 2nd-order infinitesimal quantities:

$$\frac{\partial \zeta}{\partial t} + \cancel{\frac{\partial \zeta}{\partial x} v_{ix}}\Big|_{z=\zeta} - v_{iz}\Big|_{z=\zeta} = \phi \quad \text{with } i=1,2 \quad \Rightarrow \frac{\partial \zeta}{\partial t} - v_{iz}\Big|_{z=\zeta} = \phi$$

Since v_{iz} is evaluated in $z=\zeta$ we still cannot say the expression is linear; expanding it,

$$v_{iz}(x,z,t)\Big|_{z=\zeta} = v_{iz}(x,\phi,t) + \frac{\partial v_{iz}}{\partial z}\zeta + \dots$$

So if we use v_{iz} in $z=\phi$ we have an error $\zeta\, \partial v_{iz}/\partial z$ ~ 2nd-order ∞esimal quantity that can be neglected within our 1st-order approximation. Hence the conditions read

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} - v_{1z}\Big|_{z=\phi} = \phi \\ \dfrac{\partial \zeta}{\partial t} - v_{2z}\Big|_{z=\phi} = \phi \end{cases} \quad \text{or equivalently} \quad \begin{cases} \dfrac{\partial \zeta}{\partial t} - \dfrac{\partial \varphi_1}{\partial z}\Big|_{z=\phi} = \phi \\ \dfrac{\partial \zeta}{\partial t} - \dfrac{\partial \varphi_2}{\partial z}\Big|_{z=\phi} = \phi \end{cases}$$

that is also $\boxed{\dfrac{\partial \varphi_1}{\partial z}\Big|_{z=\phi} = \dfrac{\partial \varphi_2}{\partial z}\Big|_{z=\phi}}$

or $v_{1z}\Big|_{z=\phi} = v_{2z}\Big|_{z=\phi}$

(the normal component of velocity with respect to the interface is reduced to the vertical velocity component, i.e. normal to the equilibrium interface $z=\phi$).

Dynamic condition

The terms $|\text{grad}\varphi_i|^2$ are quadratic and can be eliminated right away. Hence the condition reads

$$\rho_1 \frac{\partial \varphi_1}{\partial t}\Big|_{z=\zeta} + \rho_1 g \zeta = \rho_2 \frac{\partial \varphi_2}{\partial t}\Big|_{z=\zeta} + \rho_2 g \zeta$$

Now we cannot simply evaluate it in $z=\phi$ instead of $z=\zeta$ because the terms $\rho_i g \zeta$ would vanish and not only it is a 1st-order term (so it cannot be eliminated) but also it is the core of a gravity wave! We can circumvent this issue by taking the total derivative D/Dt of both sides of the equation; since these expressions are evaluated at the interface they hold ∀t and thus their total derivative is zero, for both sides of course. So

$$\frac{D}{Dt}\left[\rho_1 \frac{\partial \varphi_1}{\partial t}\Big|_{z=\zeta} + \rho_1 g\zeta\right] = \rho_1 \frac{\partial^2 \varphi_1}{\partial t^2}\Big|_{z=\zeta} + \rho_1 g \frac{\partial \zeta}{\partial t} + \bar{v}_1 \cdot \text{grad}\left[\rho_1 \frac{\partial \varphi_1}{\partial t}\Big|_{z=\zeta} + \rho_1 g \zeta\right] =$$

$$\frac{D}{Dt}\left[\rho_2 \frac{\partial \varphi_2}{\partial t}\Big|_{z=\zeta} + \rho_2 g\zeta\right] = \rho_2 \frac{\partial^2 \varphi_2}{\partial t^2}\Big|_{z=\zeta} + \rho_2 g \frac{\partial \zeta}{\partial t} + \bar{v}_2 \cdot \text{grad}\left[\rho_1 \frac{\partial \varphi_2}{\partial t}\Big|_{z=\zeta} + \rho_2 g \zeta\right]$$

where we can eliminate the $\vec{v}_i \cdot \mathrm{grad}[\dots]$ terms since they are higher-order infinitesimal quantities; we end up with

$$\rho_1\left[\frac{\partial^2\varphi_1}{\partial t^2}\bigg|_{z=\zeta} + g\frac{\partial\zeta}{\partial t}\right] = \rho_2\left[\frac{\partial^2\varphi_2}{\partial t^2}\bigg|_{z=\zeta} + g\frac{\partial\zeta}{\partial t}\right]$$

and now we can evaluate this expression in $z=\phi$ safely with a higher-order error $\Rightarrow$

$$\underline{\rho_1\left[\frac{\partial^2\varphi_1}{\partial t^2}\bigg|_{z=\phi} + g\frac{\partial\zeta}{\partial t}\right] = \rho_2\left[\frac{\partial^2\varphi_2}{\partial t^2}\bigg|_{z=\phi} + g\frac{\partial\zeta}{\partial t}\right]}$$

As a last step, in order to reach a condition where only $\varphi_{1,2}$ appear we invoke the linearized kinematic conditions $\frac{\partial\zeta}{\partial t} = v_{iz}\big|_{z=\phi} = \frac{\partial\varphi_i}{\partial z}\big|_{z=\phi}$ and plug them into the expression above. Finally we have

$$\text{I.B.C.}\begin{cases} \dfrac{\partial\varphi_1}{\partial z}\bigg|_{z=\phi} = \dfrac{\partial\varphi_2}{\partial z}\bigg|_{z=\phi} & (*) \\[2ex] \rho_1\left[\dfrac{\partial^2\varphi_1}{\partial t^2}\bigg|_{z=\phi} + g\dfrac{\partial\varphi_1}{\partial z}\bigg|_{z=\phi}\right] = \rho_2\left[\dfrac{\partial^2\varphi_2}{\partial t^2}\bigg|_{z=\phi} + g\dfrac{\partial\varphi_2}{\partial z}\bigg|_{z=\phi}\right] & (**) \end{cases}$$

the complete set of interface conditions for two incompressible potential flows of immiscible ideal fluids.

Note 1: In a case like a water-air interface, with $\rho_1 \gg \rho_2$ that is almost a fluid (1) topped by vacuum, in eq. $(**)$ we have an equality between terms that differ in order of magnitude, that is to say the two square brackets must vanish separately and the i.b.c. reduce, for the single fluid (1), to

$$\boxed{\frac{\partial^2\varphi}{\partial t^2}\bigg|_{z=\phi} + g\frac{\partial\varphi}{\partial z}\bigg|_{z=\phi} = \phi}$$

Furthermore the generalized Bernoulli equation for fluid (2) states

$$\rho_2\left(\frac{\partial\varphi_2}{\partial t} + \frac{1}{2}|\mathrm{grad}\,\varphi_2|^2 + g\zeta\right) + p_2 = \text{some constant we call} = p_0$$

$$\Rightarrow p_2 - p_0 = \rho_2\left(\frac{\partial\varphi_2}{\partial t} + \underbrace{\cancel{\frac{1}{2}|\mathrm{grad}\,\varphi_2|^2}}_{\text{ignored as 2nd-order}} + g\zeta\right) \quad \text{i.e. a negligible quantity if } \rho_2 \text{ is}$$

very small $\Rightarrow$ $p_2 - p_0 \simeq \phi$ $\Rightarrow$ by continuity $\underline{p_1 = p_2 = p_0 \text{ constant pressure at the interface.}}$

Finally, inverting $\partial\varphi_1/\partial t|_{z=\phi} + g\zeta = \phi$ we can obtain the surface profiles

$$\zeta(x,t) = -\frac{1}{g}\,\partial\varphi_1/\partial t\big|_{z=\phi}$$

Note 2: In order to guarantee the assumption of incompressible potential flow, a number of conditions must be met. We have seen them in general, let us now review them in the specific framework of gravity waves.

* We deal with waves of amplitude $a \ll \lambda$ wavelength and $a \ll h$ depth of the fluid basin or any other spatial scale relevant to the specific problem*. This allows us to neglect the advective term $(\bar{v} \cdot \text{grad})\bar{v}$ in Euler's equation and to treat the flow as <u>irrotational</u>.
* We deal with waves whose propagation velocity $v_{ph} \ll c$ speed of sound;

  having τ wave period as characteristic time scale,

  " λ wavelength as characteristic spatial scale,

  $\Rightarrow v_{ph} = \lambda/\tau \ll c$ implies $\tau \gg \lambda/c$;

  with $v \sim a/\tau$ typical fluid velocity scale, where $a \ll \lambda$,

  $\Rightarrow v \sim a/\tau \ll \lambda/\tau = v_p \ll c \Rightarrow v \ll c$;

  all in all, we verify here that the <u>incompressibility conditions</u> $v \ll c$, $\tau \gg \lambda/c$ are met.

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* = A small amplitude has been requested anyway in order to linearize the matching conditions at the interface.

Gravity waves in an infinitely deep basin

We consider a fluid topped by vacuum (very light fluid; as with a water basin and an atmosphere) in an unbounded domain in x, y, y-invariance, infinite depth and an unperturbed free surface $z=\phi$. In an incompressible potential flow approximation, the problem is formulated as follows:

$$\begin{cases} \nabla^2 \varphi(x,z,t) = \phi & \leftarrow \text{Laplace equation} \\ \left.\frac{\partial^2 \varphi}{\partial t^2}\right|_{z=\phi} + g \left.\frac{\partial \varphi}{\partial z}\right|_{z=\phi} = \phi & \leftarrow \text{i.b.c. at the free surface} \end{cases}$$

Consider that while there is no bottom, there is still a form of b.c. at infinite depth; physical quantities (φ, $\bar{v}$) must be finite.

We look for monochromatic wave-like solution with propagation along x, so let us try a sketch of a solution like $\underline{\varphi(x,z,t) = \cos(kx-\omega t)\, f(z)}$

where ω is the angular frequency and $k = 2\pi/\lambda$ the wave number of the wave. The solution must lead us to a $\underline{\text{dispersion relation}}$ $\omega/k = f(k)$. We do this by plugging the test solution into the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = -k^2 \cos(kx-\omega t) f(z) + \cos(kx-\omega t) \frac{d^2 f(z)}{dz^2} = \phi$$

$$\Rightarrow \frac{d^2 f}{dz^2} - k^2 f(z) = \phi \quad \text{whose general solution is}$$

$$f(z) = Ae^{kz} + Be^{-kz} \qquad \text{but we must require } B=\phi \text{ otherwise } f(z) \xrightarrow[z\to-\infty]{} \infty$$

$$\Rightarrow f(z) = Ae^{kz} \quad \Rightarrow \varphi(x,z,t) = A\cos(kx-\omega t)e^{kz}$$

Now let us use the i.b.c. by plugging in this solution into the i.b.c. condition:

$$\left.\frac{\partial^2 \varphi}{\partial t^2}\right|_{z=\phi} = -A\omega^2 \cancel{\cos(kx-\omega t)}\underbrace{\cancel{e^{kz}}}_{1 \text{ in } z=\phi} = -g\left.\frac{\partial \varphi}{\partial z}\right|_{z=\phi} = -gAk\cancel{\cos(kx-\omega t)}\underbrace{\cancel{e^{kz}}}_{1}$$

$$\Rightarrow \boxed{\omega^2 = gk} \quad \text{dispersion relation}$$

Note that $v_{ph} = \omega/k = \sqrt{g/k} = \sqrt{g\lambda/2\pi}$ is a $\underline{\text{normal}}$ dispersion relation.

Also note that a constant A is left undetermined until we set an initial condition of some sorts. Apart from that, by knowing φ we can map the velocity field since $\bar{v} = \text{grad}\,\varphi \Rightarrow$

$v_x = \partial\varphi/\partial x = -Ak\, e^{kz} \sin(kx-\omega t)$

$v_z = \partial\varphi/\partial z = Ak\, e^{kz} \cos(kx-\omega t)$

and we can see that at any given position (x,z) the vector $\bar{v}$ rotates with constant amplitude (clockwise in the xz plane).

We can determine the trajectory of a fluid element by considering (x,z) its coordinates and (x_0, z_0) its equilibrium position. For small displacement, we approximate (x,z) with (x_0, z_0) in the expressions of v_x, v_z and perform a time integration yielding

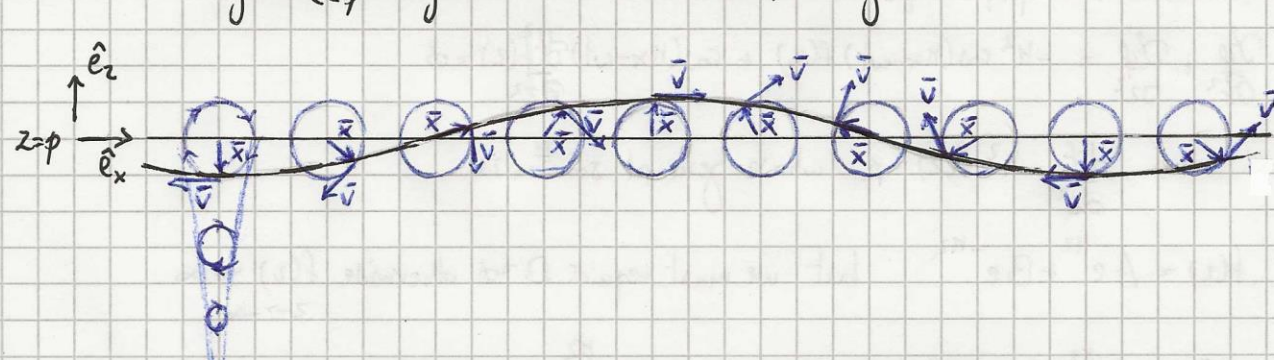
$$x - x_0 = -\frac{Ak}{\omega} e^{kz_0} \cos(kx_0-\omega t)$$

$$z - z_0 = -\frac{Ak}{\omega} e^{kz_0} \sin(kx_0-\omega t)$$

i.e. the fluid particle performs clockwise circles around the equilibrium position (x_0, z_0). Notice that these circles get exponentially smaller as the depth increases.

The wave profile is obtained as

$$\zeta(x,t) = -\frac{1}{g}\frac{\partial\varphi}{\partial t}\bigg|_{z=\phi} = -\frac{1}{g}\frac{\partial}{\partial t}\left[A\cos(kx-\omega t)e^{kz}\right]\bigg|_{z=\phi} = -\frac{1}{g}A\omega \sin(kx-\omega t)$$



While the phase velocity $v_{ph} = \omega/k = \sqrt{g/k} = \sqrt{g\lambda/2\pi}$, the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\sqrt{kg}\right) = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$

($v_g < v_f$, as per normal dispersion relation)

Gravity waves in a finite-depth basin

Now we consider the same problem; An ideal fluid in an unbounded domain in x and y, with y-invariance, unperturbed free surface $z=\phi$, in incompressible potential flow approximation; but a finite depth h. We shall then consider a wave amplitude $a \ll h$ in addition to $a \ll \lambda$; furthermore, a free-slip boundary condition at the bottom $z=-h$ (no penetration into the solid bottom). Hence the problem is stated as follows:

$$\begin{cases} \nabla^2 \varphi = \phi \\ \left.\dfrac{\partial \varphi}{\partial z}\right|_{z=-h} = \phi \qquad (\text{i.e. } v_z(x,-h,t) = \phi) \\ \left.\dfrac{\partial^2 \varphi}{\partial t^2}\right|_{z=\phi} + g \left.\dfrac{\partial \varphi}{\partial z}\right|_{z=\phi} = \phi \end{cases}$$

Once again we sketch a test solution in the form of a monochromatic wave propagating along x with angular frequency ω and wave number k; using the Laplace equation and the b.c. we shall obtain the specific solution and dispersion relation.

Plugging the test solution $\varphi(x,z,t) = \cos(kx-\omega t) f(z)$ into the Laplace equation,

$$\nabla^2 \varphi = -k^2 \cos(kx-\omega t) f(z) + \cos(kx-\omega t)\, d^2 f/dz^2 = \phi$$

$$\Rightarrow \frac{d^2 f(z)}{dz^2} - k^2 f(z) = \phi \qquad \text{has solution} \quad f(z) = Ae^{kz} + Be^{-kz}$$

$$\Rightarrow \varphi(x,z,t) = \left(Ae^{kz} + Be^{-kz}\right) \cos(kx-\omega t)$$

The b.c. at the bottom requires

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=-h} = \left(Ak e^{-kh} - Bk e^{kh}\right) \cos(kx-\omega t) = \phi \iff \underline{\underline{Ae^{-kh} - Be^{kh} = \phi}}$$

The i.b.c. at the free surface requires

$$-\omega^2 \left(Ae^{kz} + Be^{-kz}\right)\Big|_{z=\phi} \cos(kx-\omega t) + gk\left(Ae^{kz} - Be^{-kz}\right)\Big|_{z=\phi} \cos(kx-\omega t) = \phi$$

$$\Rightarrow -\omega^2 (A+B) + gk(A-B) = \phi \quad \text{i.e.} \quad \underline{\underline{A(kg-\omega^2) - B(kg+\omega^2) = \phi}}$$

The full set of b.c. + i.b.c. yields a system of two equations for the two unknown constants A, B:

$$\begin{cases} e^{-kh} A - e^{kh} B = \phi \\ (kg-\omega^2) A - (kg+\omega^2) B = \phi \end{cases} \quad \text{i.e. a } \underline{\text{homogeneous linear system}} \ \underline{\underline{C}}\, \bar{X} = \phi$$

with coefficient matrix $\underline{\underline{C}} = \begin{pmatrix} e^{-kh} & e^{kh} \\ kg-\omega^2 & -(kg+\omega^2) \end{pmatrix}$ and unknown vector $\bar{X} = \begin{pmatrix} A \\ B \end{pmatrix}$.

Excluding the trivial solution $\bar{X} = \bar{\phi}$ ($A = B = \phi$), a homogeneous linear system has an infinite number of solutions obtained by setting $\boxed{\det \underline{\underline{C}} = \phi}$. Infinite solutions means here that one constant (say A) is arbitrary and the second one (B) is determined as a function of A; the latter is determined by means of another constraint, like an initial condition.

Notice that we have cast the problem in a forms similar to that of an eigenvalue problem $(\underline{\underline{A}} - \lambda \underline{\underline{1}})\bar{X} = \phi$.

So let us set $\det \underline{\underline{C}} = \phi$:

$$\det \underline{\underline{C}} = -e^{-kh}(kg+\omega^2) + e^{kh}(kg-\omega^2) = \phi$$

$$\Rightarrow -\omega^2(e^{-kh}+e^{kh}) + kg(e^{kh}-e^{-kh}) = \phi$$

$$\Rightarrow \omega^2 = kg\,\frac{e^{kh}-e^{-kh}}{e^{kh}+e^{kh}} = kg\tanh(kh)$$

The dispersion relation reads $\boxed{\omega^2 = kg\tanh(kh)}$.

Since B is in the end to be expressed as a function of A, from

$$Ae^{-kh} - Be^{kh} = \phi \quad \Rightarrow \quad B = Ae^{-2kh}$$

let us rewrite the constants using a single one; we define

$$A \doteq e^{kh}\, C/2 \quad \Rightarrow \quad B \doteq e^{-kh}\, C/2$$

$$\Rightarrow f(z) = Ae^{kz} + Be^{-kz} = \frac{C}{2}e^{k(z+h)} + \frac{C}{2}e^{-k(z+h)} = C\cosh[k(z+h)]$$

So we finally have the velocity potential

$$\boxed{\varphi(x,z,t) = C\cosh[k(z+h)]\cos(kx-\omega t)}$$

The wave profile is $\zeta(x,t) = -\frac{1}{g}\frac{\partial\varphi}{\partial t}\Big|_{z=\phi} = -\frac{C\omega}{g}\cosh(kh)\sin(kx-\omega t)$

The phase velocity is $v_{ph} = \frac{\omega}{k} = \sqrt{\frac{g\tanh(kh)}{k}}$ ($\tanh(x)$ is less than linear $\Rightarrow$ normal dispersion relation)

The group velocity is $v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left[(kg\tanh(kh))^{\frac{1}{2}}\right] =$

$$= \frac{1}{2}\left[gk\tanh(kh)\right]^{-\frac{1}{2}}\left[g\tanh(kh) + gkh\,\frac{1}{\cosh^2(kh)}\right] =$$

$$= \frac{1}{2}\left(\frac{g}{k\tanh(kh)}\right)^{\frac{1}{2}}\left[\tanh(kh) + \frac{kh}{\cosh^2(kh)}\right] = \frac{1}{2}\underbrace{\left(\frac{g\tanh(kh)}{k}\right)^{\frac{1}{2}}}_{G = \omega/k = v_{ph}}\left[1 + \frac{kh}{\sinh(kh)\cosh(kh)}\right] \Rightarrow$$

$$v_g = \frac{1}{2}\,\frac{\omega}{k}\left[1 + \frac{2kh}{\sinh(2kh)}\right]$$

These general expressions involving hyperbolic functions may look more or less complicated; it is interesting to explore asymptotic cases to get a better grasp of the physics occurring in this phenomenon.

⊙ $kh \gg 1$, i.e. $\lambda \ll h$ (SHORT WAVES with respect to the basin's depth)

$$\lim_{kh \to +\infty} \tanh(kh) = \lim_{kh \to +\infty} \frac{e^{kh} - \cancel{e^{-kh}}}{e^{kh} + \cancel{e^{-kh}}} \simeq e^{kh}/e^{kh} = 1 \Rightarrow \boxed{v_{ph} = \sqrt{\frac{g\tanh(kh)}{k}} = \sqrt{g/k}}$$

$$\lim_{kh \to +\infty} \sinh(2kh) \simeq \frac{e^{2kh}}{2} \Rightarrow \lim_{kh \to +\infty} 2kh/\sinh(2kh) \simeq 4kh/e^{2kh} = \phi$$

$$\Rightarrow \boxed{v_g = \frac{1}{2}\,\frac{\omega}{k} = \frac{1}{2}v_{ph} = \frac{1}{2}\sqrt{\frac{g}{k}}}$$ as in the infinite-depth case

(indeed here $kh \gg 1 \Rightarrow h$ very large with respect to wavelength)

⊙ $kh \ll 1$, i.e. $\lambda \gg h$ (LONG WAVES or SHALLOW basin)

$$\lim_{kh \to \phi} \tanh(kh) = \phi$$ but the expansion $\tanh(kh) = kh - (kh)^3/3 + \ldots \simeq kh$

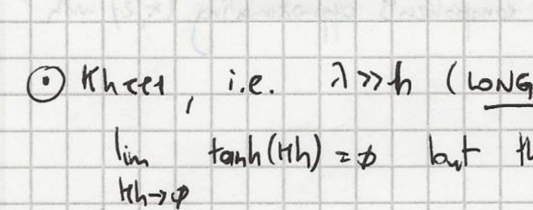
$$\Rightarrow v_{ph} = \sqrt{\frac{g\tanh(kh)}{k}} = \sqrt{gh}$$

Similarly, $\sinh(2kh) \simeq 2kh + (2kh)^3/3! + \ldots \simeq 2kh$

$$\Rightarrow v_g = \frac{1}{2}\,\frac{\omega}{k}\left[1 + \frac{2kh}{2kh}\right] = \frac{\omega}{k} = v_{ph} = \sqrt{gh} \Rightarrow \boxed{v_{ph} = v_g = \sqrt{gh}}$$

We have a trivial dispersion relation: Velocities are independent of the wavelength.

Qualitatively we could sketch a $v_{ph}(\lambda)$ (logarithmic) diagram as follows:



v_{ph} saturates at large wavelengths, while it scales with $\sqrt{\lambda}$ at small ones; and notice that v_{ph} is also a monotonically increasing function of depth h at intermediate and large λ, while it is h-independent at small λ.

Let us see the fluid element trajectory in this case. The potential $\varphi(x,z,t) = C\cosh(K(z+h))\cos(Kx-\omega t)$

we have $\quad v_x = \partial_x\varphi = -KC\cosh[K(z+h)]\sin(Kx-\omega t)$

$\quad v_z = \partial_z\varphi = KC\sinh[K(z+h)]\cos(Kx-\omega t)$

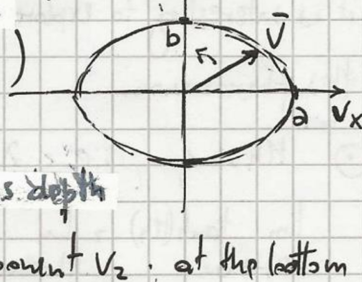
indicating a clockwise rotation of $\vec{v}$; we can understand the behaviour of the amplitude writing

$$\frac{v_x^2}{(KC)^2\cosh^2[K(z+h)]} + \frac{v_z^2}{(KC)^2\sinh^2[K(z+h)]} = \sin^2(Kx-\omega t) + \cos^2(Kx-\omega t) = 1$$

and with $a \doteq KC\cosh(K(z+h))$

$b \doteq KC\sinh[K(z+h)]$ $\quad \left(a > b \;\; \forall z \in [-h;\phi] ; \; \lim_{z\to -h} a = KC , \; \lim_{z\to -h} b = \phi \right)$

we have an ellipse $\quad \frac{v_x^2}{a^2} + \frac{v_z^2}{b^2} = 1$



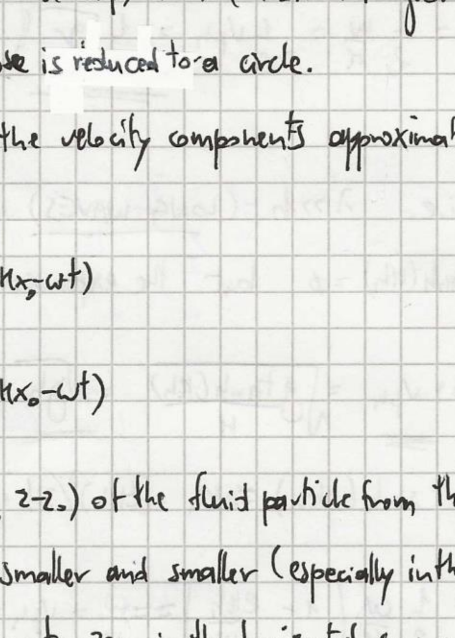
The velocity vector performs rotations with elliptical shape; as depth increases, the ellipse gets smaller, especially in the vertical component v_z; at the bottom $z=-h$ the ellipse has zero vertical dimension, i.e. $v_z = \phi$ (indeed in compliance with the b.c.) while a horizontal velocity $v_x \neq \phi$ is accepted (free slip). Notice that for great depth $h\to\infty$ $\cosh \sim \sinh \sim \exp$ and $a \simeq b$, i.e. an ellipse is reduced to a circle.

To get the trajectory we integrate again the velocity components approximating (x,z) with the equilibrium position (x_0, y_0); hence

$$x - x_0 = -\frac{KC}{\omega}\cosh[K(h+z_0)]\cos(Kx_0-\omega t)$$

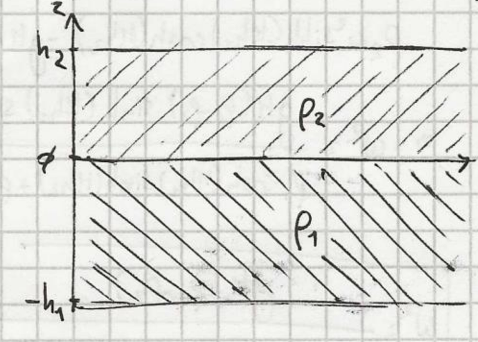
$$z - z_0 = -\frac{KC}{\omega}\sinh[K(h+z_0)]\sin(Kx_0-\omega t)$$

which shows again a displacement $(x-x_0, z-z_0)$ of the fluid particle from the equilibrium position that is elliptical in shape and gets smaller and smaller (especially in the vertical direction) as the depth increases – without reducing to zero in the horizontal component: There is a horizontal oscillation on the bottom.



Gravity waves at the interface between two fluid layers

We consider here two ideal fluids in incompressible potential flow approximation: A fluid ① with density ρ_1 and depth h_1 ($z \in [-h_1, 0]$) topped by a fluid ② with density ρ_2 and depth h_2 ($z \in [0, h_2]$). The unperturbed interface is at $z=0$ and the layers are unbounded in x and y, with y-invariance. The problem is stated as follows:



$$
\begin{cases}
\nabla^2 \varphi_1 = 0 \\
\nabla^2 \varphi_2 = 0 \\
v_{1z}(x, -h_1, t) = \left.\dfrac{\partial \varphi_1}{\partial z}\right|_{z=-h_1} = 0 \\
v_{2z}(x, h_2, t) = \left.\dfrac{\partial \varphi_2}{\partial z}\right|_{z=h_2} = 0 \\
v_{1z}(x, 0, t) = v_{2z}(x, 0, t) = v_z \qquad \left(\left.\dfrac{\partial \varphi_1}{\partial z}\right|_{z=0} = \left.\dfrac{\partial \varphi_2}{\partial z}\right|_{z=0} \right) \\
\rho_1 \left.\dfrac{\partial^2 \varphi_1}{\partial t^2}\right|_{z=0} + \rho_1 g \underbrace{\left.\dfrac{\partial \varphi_1}{\partial z}\right|_{z=0}} = \rho_2 \left.\dfrac{\partial^2 \varphi_2}{\partial t^2}\right|_{z=0} + \rho_2 g \underbrace{\left.\dfrac{\partial \varphi_1}{\partial z}\right|_{z=0}}
\end{cases}
$$

$\longrightarrow$ both equal to $v_z(z=0)$ so both $\partial\varphi_1/\partial z$ or $\partial\varphi_2/\partial z$ $\Rightarrow$ rewritten as

$$
\left.\frac{\partial \varphi_1}{\partial z}\right|_{z=0} = \frac{1}{g(\rho_1-\rho_2)} \left(\rho_2 \left.\frac{\partial^2 \varphi_2}{\partial t^2}\right|_0 - \rho_1 \left.\frac{\partial^2 \varphi_1}{\partial t^2}\right|_0 \right)
$$

We have seen that the solution of the problem with a single finite-depth fluid has a decomposition with a $f(z) \sim \cosh[K(z-h)]$ so, by exploiting the b.c. at the solid boundaries we expect here solutions in a similar form, i.e.

$$\varphi_1(x,z,t) = A\cosh[K(z+h_1)] \cos(Kx-\omega t)$$

$$\varphi_2(x,z,t) = B\cosh[K(z-h_2)] \cos(Kx-\omega t)$$

and we apply the Kinematic b.c. in $z=0$;

$$AK \sinh(Kh_1)\cos(Kx-\omega t) = BK\sinh(-Kh_2)\cos(Kx-\omega t) = -BK\sinh(Kh_2)\cos(Kx-\omega t)$$

$$\Rightarrow \underline{A\sinh(Kh_1) + B\sinh(Kh_2) = 0}$$

and then the dynamic condition

$$g(\rho_1-\rho_2)AK\sinh(Kh_1)\cancel{\cos(Kx-\omega t)} = -\rho_2\omega^2 B \overbrace{\cosh(-Kh_2)}^{=\cosh(Kh_2)}\cancel{\cos(Kx-\omega t)} + \rho_1\omega^2 A\cosh(Kh_1)\cancel{\cos(Kx-\omega t)}$$

$$\Rightarrow \underline{[gK(\rho_1-\rho_2)\sinh(Kh_1) - \rho_1\omega^2\cosh(Kh_1)]A + \rho_2\omega^2\cosh(Kh_2)B = 0}$$

The two condition collectively represent again a homogeneous linear system in A and B:

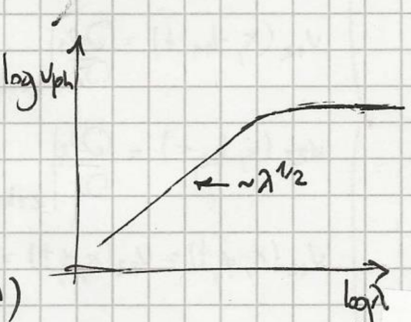
$$\begin{cases} \sinh(kh_1)A + \sinh(kh_2)B = 0 \\ [gk(\rho_1-\rho_2)\sinh(kh_1) - \rho_1\omega^2\cosh(kh_1)]A + \rho_2\omega^2\cosh(kh_2)B = 0 \end{cases} \Rightarrow \underline{\underline{C}}\,\bar{x} = 0, \quad \bar{x} = \begin{pmatrix} A \\ B \end{pmatrix}$$

so the non-trivial solutions are determined by setting the determinant of the coefficient matrix $\underline{\underline{C}}$ equal to zero:

$$\rho_2\omega^2\sinh(kh_1)\cosh(kh_2) - gk(\rho_1-\rho_2)\sinh(kh_1)\sinh(kh_2) + \rho_1\omega^2\cosh(kh_1)\sinh(kh_2) = 0$$

$$\Rightarrow \omega^2 = \frac{gk(\rho_1-\rho_2)\sinh(kh_1)\sinh(kh_2)}{\rho_1\cosh(kh_1)\sinh(kh_2) + \rho_2\sinh(kh_1)\cosh(kh_2)}$$ and dividing numerator and denominator by $\sinh(kh_1)\sinh(kh_2)$

$$\boxed{\omega^2 = \frac{gk(\rho_1-\rho_2)}{\rho_1\coth(kh_1) + \rho_2\coth(kh_2)}}$$ dispersion relation



⊙ With $kh_1 \gg 1$, $kh_2 \gg 1$ (both layers are deep with respect to λ)

$$\coth(x) = \cosh(x)/\sinh(x) = 1/\tanh(x)$$

and $\lim_{x\to+\infty} 1/\tanh(x) = 1$

$$\Rightarrow \boxed{\omega^2 = gk\,\frac{\rho_1-\rho_2}{\rho_1+\rho_2}} \Rightarrow v_{ph} = \omega/k \propto 1/\sqrt{k} \propto \sqrt{\lambda}$$ normal dispersion relation

(notice that $\omega^2 \geq 0$ and we must have $\rho_1 > \rho_2$, otherwise, we shall later see that an instability occurs if the fluid on top is heavier)

⊙ $kh_1 \ll 1$, $kh_2 \ll 1$ (shallow layers)

$$\lim_{x\to 0}\coth(x) = \lim_{x\to 0} 1/\tanh x \simeq 1/x$$

$$\Rightarrow \omega^2 = \frac{gk(\rho_1-\rho_2)}{\rho_1/kh_1 + \rho_2/kh_2} \Rightarrow \boxed{\omega^2 = gk^2\,\frac{(\rho_1-\rho_2)h_1h_2}{\rho_1h_2+\rho_2h_1}} \Rightarrow v_{ph} = \left(\frac{g(\rho_1-\rho_2)h_1h_2}{\rho_1h_2+\rho_2h_1}\right)^{\frac{1}{2}}$$

trivial dispersion relation