Energy transport in gravity waves

Let us consider a gravity wave propagating along the x direction in an ideal fluid, within an incompressible potential flow approximation. The system is y-invariant, z vertical axis, and we know

$$\left.\begin{aligned} \varphi(x,z,t) &= A\cosh[k(z+h)]\cos(kx-\omega t) \\ \zeta(x,t) &= -\frac{1}{g}\omega A\cosh(kh)\sin(kx-\omega t) \end{aligned}\right\} \text{for a wave in a basin with bottom } z=-h \text{ and vacuum (negligible fluid) on top}$$

Let us consider a surface through which we want to calculate the energy flux. This is a plane with normal unit vector $\hat{e}_x$, width L in the y-direction, and height measured between the bottom $z=-h$ to the top of the fluid layer $z=\zeta$; let us call this surface $S(\zeta)$. When we studied the energy flux we found out the flux density is $\rho(\varepsilon_m+w)\bar{v}$, that is here $\rho(\frac{1}{2}v^2+gz+p/\rho+\varepsilon)\bar{v}$ and by ignoring ε (conserved in an incompressible flow) the flux through $S(\zeta)$ results

$$Q^{\varepsilon}(S) = \int_S \rho\left(\frac{1}{2}v^2+gz+p/\rho\right)\bar{v}\cdot d\bar{S} = \rho\int_S\left(\frac{1}{2}v^2+gz+p/\rho\right)v_x\,dy\,dz$$

Making use of the generalized Bernoulli's equation applied between a generic z and ζ,

$$\frac{1}{2}v^2+\frac{p}{\rho}+gz+\frac{\partial\varphi}{\partial t} = \frac{1}{2}v^2(\zeta)+\frac{p_0}{\rho}+g\zeta+\frac{\partial\varphi(\zeta)}{\partial t},$$

pressure at the free surface is a constant which we can set as $p_0=\phi$ (negligible for very light upper fluid); the kinetic terms are quadratic, hence negligible in a linear approximation; finally, the interface boundary condition yields

$$g\zeta(x,t)+\frac{\partial}{\partial t}\varphi(x,\phi,t)=\phi$$ and therefore we simplify the equation above into

$$\frac{1}{2}v^2+\frac{p}{\rho}+gz=-\frac{\partial\varphi}{\partial t}$$

$$\Rightarrow Q^{\varepsilon}(S) = -\int_y^{y+L}\int_{z=-h}^{z=\zeta}\rho\frac{\partial\varphi}{\partial t}v_x\,dy\,dz = -\rho L\int_{z=-h}^{z=\zeta}\frac{\partial\varphi}{\partial t}v_x\,dz = -\rho L\int_{-h}^{\zeta}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x}\,dz$$

Now let us split the integral over two subintervals $[-h,\phi]$ and $[\phi,\zeta]$; the second term, discarding higher-order infinitesimal contribution, can be approximated using the value in $z=\phi$ of the integrand, so that

$$Q^{\varepsilon}(S) = -\rho L\int_{-h}^{\phi}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x}\,dz - \rho L\frac{\partial\varphi}{\partial t}(x,\phi,t)\frac{\partial\varphi}{\partial x}(x,\phi,t)\,\zeta(x,t)$$

The first term is second order in magnitude, while the second one is a third-order contribution. Furthermore, we are interested in calculating a period-average quantity, and one could see that the second term, as a product of three sinusoidal function, has a zero average. So let us make a time average over a period T:

$$\langle Q^{E}(S)\rangle = -\rho L \frac{1}{T}\int_{t_0}^{t_0+T}\int_{z=-h}^{z=\phi} \frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x}\,dz\,dt$$ and plugging in the explicit form of φ

$$\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x} = \omega A\cosh[K(z+h)]\sin(Kx-\omega t)\cdot(-K)A\cosh[K(z+h)]\sin(Kx-\omega t) =$$
$$= -K\omega A^2\cosh^2[K(z+h)]\sin^2(Kx-\omega t)$$

and since $\langle \sin(Kx-\omega t)\rangle = \frac{1}{2}$, $\quad \int\cosh^2(\alpha x)\,dx = \frac{1}{4\alpha}\sinh(2\alpha x) + x/2 + C$

$$\langle Q^{E}(S)\rangle = \frac{1}{2}\rho L K\omega A^2\int_{-h}^{\phi}\cosh^2(K(z+h))\,dz = \frac{1}{2}\rho L K\omega A^2\left[\frac{1}{4K}\sinh(2K(z+h)) + \frac{z+h}{2}\right]_{-h}^{\phi} =$$
$$= \frac{1}{8}\rho L\omega A^2\left[2Kh + \sinh(2Kh)\right]$$

Now let us calculate the time-averaged amount of mechanical energy stored in a region $R(\zeta)$ that is just upstream of $S(\zeta)$ and extends in the x direction of a quantity $\Delta x = \lambda$ (one wavelength). Any point within $R(\zeta)$ is a point (x,y,z) such that

$$\begin{cases} x_0 < x < x_0+\lambda \\ y_0 < y < y_0+L \\ -h < z < \zeta(x,t) \end{cases}$$

We also call $R(\phi)$ a subregion of $R(\zeta)$ of parallelepipedal shape obtained considering only the part with $z \in [-h,\phi]$. the boundary surface of $R(\phi)$ shall be called $S_R(\phi)$ and the top surface of $S_R(\phi)$ (points $(x,y,z=\phi)$) shall be $A(\phi)$.

The mechanical energy in $R(\zeta)$ is now written substracting from it the energy in the absence of the wave (when the region occupied by the fluid at rest is just $R(\phi)$):

$$E_m(R(\zeta)) = \int_{R(\zeta)}\rho\left(\frac{1}{2}v^2+gz\right)d^3x - \int_{R(\phi)}\rho g z\,d^3x$$

In the first term, the region above $z=\phi$ contributes an amount of energy that is approximated (neglecting higher-order infinitesimal terms) with the integrand value in $z=\phi$, then integrating in x and y over $A(\phi)$ and multiplying by $\zeta(x,t)$; the potential energy contribution over the region $R(\phi)$ is also subtracted and we get

$$E_m(R(\zeta)) = \int_{R(\phi)} \rho \frac{1}{2} v^2 d^3x + \int_{A(\phi)} \rho \frac{1}{2} v^2 \zeta(x,t)\, dx dy + \int_{A(\phi)} \underbrace{\left(\int_{z=\phi}^{z=\zeta} \rho g z \, dz \right)}_{\to \frac{1}{2}\zeta^2(x,t)} dx dy \quad ;$$

once again the second term is not only negligible as a higher-order infinitesimal, but made of oscillating functions whose overall result is a vanishing time average, so

$$E_m(R(\zeta)) = \frac{1}{2}\rho \int_{R(\phi)} v^2 d^3x + \frac{1}{2}\rho g \int_{A(\phi)} \zeta^2(x,t)\, dx dy$$

We shall now use some algebra to manipulate the first integral; indeed $v^2 = |\mathrm{grad}\varphi|^2$ and we can notice that $\mathrm{div}(\varphi\, \mathrm{grad}(\varphi)) = \mathrm{grad}\varphi \cdot \mathrm{grad}\varphi + \varphi \underbrace{\nabla^2\varphi}_{=\phi} = |\mathrm{grad}\varphi|^2 = v^2$

$$\Rightarrow E_m(R(\zeta)) = \frac{1}{2}\rho \int_{R(\phi)} \mathrm{div}(\varphi\, \mathrm{grad}\varphi)\, d^3x + \frac{1}{2}\rho g \int_{A(\phi)} \zeta^2(x,t)\, dx dy =$$

$$= \frac{1}{2}\rho \int_{S_R(\phi)} \varphi\, \mathrm{grad}\varphi \cdot \hat{n}\, da + \frac{1}{2}\rho g \int_{A(\phi)} \zeta^2 dx dy$$

We can cut out some parts of the flux integral over $S_R(\phi)$, and precisely where $\vec{v} = \mathrm{grad}\varphi$ has a zero normal component with respect to the integration surface, or where the net flux is zero:

- At the bottom, $z = -h$, the b.c. requires $v_z = \mathrm{grad}\varphi \cdot \hat{e}_z = \phi$;
- Due to y-invariance, v_y = constant and thus on the two surfaces $y = y_0$, $y = y_0 + L$ of the volume $R(\phi)$ the incoming and outgoing fluxes are equal and balance out;
- Periodicity along x requires incoming and outgoing fluxes through the surfaces at $x = x_0$, $x = x_0 + L$ to be equal and cancel out again.

What is left is just the flux through $A(\phi)$, in the vertical direction; so

$$E_m(R(\zeta)) = \frac{1}{2}\rho \int_{A(\phi)} \varphi(x,\phi,t) \left.\frac{\partial\varphi}{\partial z}\right|_{z=\phi} dx dy + \frac{1}{2}\rho g \int_{A(\phi)} \zeta^2(x,t)\, dx dy =$$

recalling $\zeta = -\frac{1}{g} \left.\frac{\partial\varphi}{\partial t}\right|_{z=\phi}$

$$= \frac{1}{2}\rho \int_{A(\phi)} \left[\varphi(x,\phi,t)\frac{\partial \varphi}{\partial z}\bigg|_{z=\phi} + \frac{1}{g}\left(\frac{\partial \varphi}{\partial t}\right)^2\bigg|_{z=\phi} \right] dx\,dy$$

Let us rewrite explicitly φ and its derivatives before we perform the time average:

$$\varphi(x,z,t) = A\cosh[k(z+h)]\cos(kx-\omega t) \rightarrow \varphi(x,\phi,t) = A\cosh(kh)\cos(kx-\omega t)$$

$$\frac{\partial \varphi}{\partial z}\bigg|_{z=\phi} = Ak\sinh(kh)\cos(kx-\omega t)$$

$$\frac{\partial \varphi}{\partial t}\bigg|_{z=\phi} = A\omega\cosh(kh)\sin(kx-\omega t)$$

$$\Rightarrow \odot\ \frac{1}{T}\int_T \varphi(x,\phi,t)\frac{\partial \varphi}{\partial z}\bigg|_{z=\phi} dt = \frac{1}{T}\int_T A^2 k \underbrace{\sinh(kh)\cosh(kh)}_{= \frac{1}{2}\sinh(2kh)} \cos^2(kx-\omega t)\,dt = \frac{1}{4}A^2 k\sinh(2kh)$$

$$\odot\ \frac{1}{T}\int_T \frac{1}{g}A^2\omega^2\cosh^2(kh)\sin^2(kx-\omega t)\,dt = \frac{1}{2g}A^2\omega^2\cosh^2(kh) = \quad \text{using } \omega^2 = gk\tanh(kh)$$

$$= \frac{1}{2}A^2 k\sinh(kh)\cosh(kh) = \frac{1}{4}A^2 k\sinh(2kh)$$

We find here that the two terms are equal. This should not really come as a surprise: It is a more general fact that for small oscillations, the time averages of kinetic and potential energy are the same. So we finally end up writing

$$\langle E_m(R(\zeta))\rangle = \frac{1}{2}\rho\int_{A(\phi)} \frac{1}{2}A^2 k\sinh(2kh)\,dx\,dy = \frac{1}{4}\rho A^2 k\sinh(2kh)\cdot\lambda L \quad \text{with } \lambda = 2\pi/k$$

$$\Rightarrow \langle E_m(R(\zeta))\rangle = \frac{1}{2}\pi\rho L A^2\sinh(2kh)$$

In general terms, we can write the flux of energy multiplying the energy density per unit volume ρ_{en} times the flux velocity v_f and the area Σ of the cross section S through which we calculate the flux; inverting the expression, we can get v_f as follows:

$$\langle Q^E(S)\rangle = \rho_{en}\Sigma v_f = \rho_{en}\frac{\overset{\text{volume}}{\Sigma\lambda}}{\lambda}v_f = \frac{\langle E_m(R(\zeta))\rangle}{\lambda}v_f$$

$$\Rightarrow v_f = \lambda\langle Q^E(S)\rangle / \langle E_m(R(\zeta))\rangle = \lambda\frac{1}{8}\rho L\omega A^2[2kh+\sinh(2kh)] \Big/ \frac{1}{2}\pi\rho L A^2\sinh(2kh) =$$

$$= \frac{\lambda\omega}{4\pi}\left[1+\frac{2kh}{\sinh(2kh)}\right] = \frac{\omega}{2k}\left[1+\frac{2kh}{\sinh(2kh)}\right] = v_g$$

$v_f = v_g$ group velocity tells us that energy is not propagated at phase velocity, but at group velocity (even for a monochromatic wave*). Thus for gravity waves the deepest physical meaning of the group velocity is that of mean energy transport velocity (even more than that of translation velocity of a wave packet, which is always undergoing a progressive deformation in a dispersive medium).

[* = The carrier wavelength will be that of the wave itself.]

Appendix - Phase and group velocity (a very brief summary)

A monochromatic (single-wavelength) wave featuring an angular frequency ω and wavenumber k propagates with a velocity called PHASE VELOCITY $\underline{v_{ph} = \omega/k}$ (with $k = 2\pi/\lambda$, λ wavelength). This ω/k ratio does not necessarily have to be constant; on the contrary, v_{ph} and ω can be a function of k. The expression $\omega(k)/k$ is called DISPERSION RELATION (the name comes from the fact that propagation in a "dispersive medium" will yield different velocities for waves with different wavelength/wavenumber, hence a "spreading" of a wavepacket made out of waves with a range of k values).

We can say that the dispersion relation is

NORMAL: v_f is an increasing function of increasing λ

ANOMALOUS: v_f is a decreasing function of increasing λ

TRIVIAL: $v_f = \omega/k$ = constant (ω is a constant; see, e.g., d'Alembert's wave equation, for electromagnetic waves in vacuum)

Whatever dispersion relation we have, a monochromatic wave with angular frequency ω and wavenumber k can be expressed as

$$f(x,t) = A\exp[i(kx-\omega t)] \; A\exp[ik(x-v_{ph}t)] = g(x-v_{ph}t)$$

for a simple, one-dimensional case; a more general solution to a wave equation is a whole superposition of monochromatic waves; in a continuous limit, this is an integral

$$F(x,t) = \int_{-\infty}^{+\infty} f(k)\exp[i(kx-\omega(k)t)]\,dk \qquad \text{i.e. a } \underline{\text{wavepacket}}$$

with $f(k)$ amplitude (a function that must be such that the integral is acceptable). A physically meaningful and interesting case is a superposition of waves within a limited range of k, i.e. $f(k) = \phi$ outside a certain interval of k values and a peak amplitude within that range corresponding to a value $k = k_0$. The wave corresponding to such k_0 has maximum amplitude and is called CARRIER WAVE of the wavepacket. We now want to evaluate the propagation velocity of the wavepacket as a whole.

Since $f(k) \neq \phi$ only over a limited k range, $\omega(k)$ can be expanded to first order:

$$\omega(k) \simeq \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k=k_0}(k-k_0) = \omega(k_0) + \omega_0'(k-k_0)$$

where we called $\omega_0' = \left.\frac{d\omega}{dk}\right|_{k=k_0}$

$$\Rightarrow F(x,t) = \int_{-\infty}^{+\infty} f(k) \exp[i(kx-\omega(k)t)]\, dk = \int_{k_1}^{k_2} f(k) \exp[i(kx-\omega(k)t)]\, dk =$$

(where $[k_1, k_2]$ interval / $f(k)=\phi \ \forall k \notin (k_1, k_2)$; $k_0 \in [k_1, k_2]$; $f(k_0) = \max(f(k))$)

$$= \int_{k_1}^{k_2} f(k) \exp\{ i \underbrace{[kx - (\omega_0 + (k-k_0)\omega_0')t]}_{kx(-k_0x + k_0x) - \omega_0 t - (k-k_0)\omega_0' t = (k-k_0)(x-\omega_0' t) + k_0 x - \omega_0 t} \}\, dk =$$

$$= \int_{k_1}^{k_2} f(k) \exp\{ i[(k-k_0)(x-\omega_0' t) + \underbrace{k_0 x - \omega_0 t}_{\text{a constant with respect to } k}]\}\, dk =$$

$$= \exp[i(k_0 x - \omega_0 t)] \int_{k_1}^{k_2} f(k) \exp\{ i[(k-k_0)(x-\omega_0' t)]\}\, dk = \quad \text{with } \eta \doteq k - k_0$$

$$= \exp(i(k_0 x - \omega_0 t)) \int_{k_1-k_0}^{k_2-k_0} f(\eta + k_0) \exp[i\eta(x-\omega_0' t)]\, d\eta =$$

$$= \exp[i(k_0 x - \omega_0 t)]\; g(x - \omega_0' t)$$

which represents a carrier wave $\exp[i(k_0x-\omega_0 t)]$ carrying the wavepacket with an amplitude $g(x-\omega_0' t)$; the translation velocity of this amplitude is ω_0', and we call it

$$\boxed{v_g = \omega_0' = \left.\frac{d\omega}{dk}\right|_{k=k_0}} \qquad \text{GROUP VELOCITY}$$

The propagation is not "painless": If in the expansion we had considered higher-order terms, in the calculation we would have found a slow deformation (dispersion) of the wavepacket. In other words, the wavepacket does not simply undergo a rigid translation.

The occurrence of a normal/anomalous dispersion relation impacts the value of v_g with respect to that of v_{ph}:

$$\frac{d}{dk} v_{ph} = \frac{d}{dk}\left(\frac{\omega}{k}\right) = \frac{1}{k^2}\left(\frac{d\omega}{dk} k - \omega\right) = \frac{1}{k^2}(v_g k - \omega) \gtrless \phi$$

$$\Leftrightarrow v_g k - \omega \gtrless \phi \quad , \text{ i.e. } v_g \gtrless \omega/k = v_{ph}$$

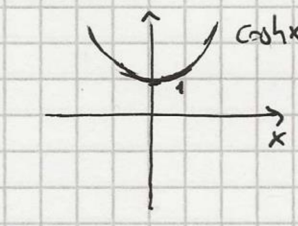
So $\quad v_g < v_{ph} \iff \frac{dv_{ph}}{dk} < \phi$ i.e. $\frac{dv_{ph}}{d\lambda} > \phi$ $\qquad$ normal dispersion relation

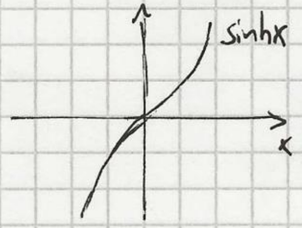
$\qquad v_g > v_{ph} \iff \frac{dv_{ph}}{dk} > \phi$ i.e. $\frac{dv_{ph}}{d\lambda} < \phi$ $\qquad$ anomalous dispersion relation

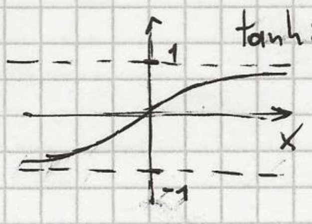
Appendix – A digest of hyperbolic functions

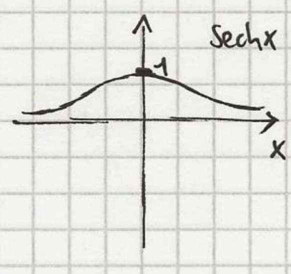
$$\cosh x = \frac{e^x + e^{-x}}{2} \; ; \; \sinh x = \frac{e^x - e^{-x}}{2} \; ; \; \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

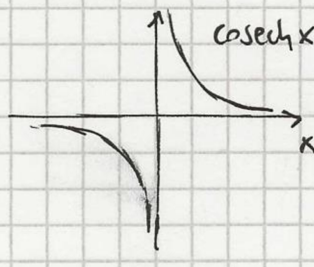
$$\operatorname{sech} x = 1/\cosh x \; ; \; \operatorname{cosech} x = 1/\sinh x \;\; \forall x \neq 0 \; ; \; \coth x = 1/\tanh x \;\; \forall x \neq 0$$

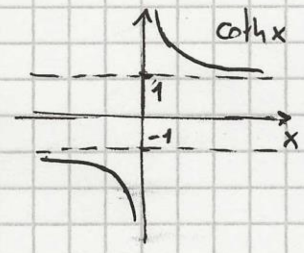












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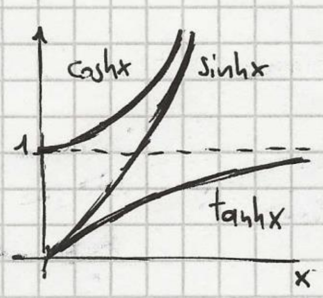

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx}(\cosh x) = \sinh x \; ; \; \frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Double angle formulae: $\cosh(2x) = \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

$$\sinh(2x) = 2\sinh x \cosh x$$

$$\tanh(2x) = 2\tanh x / (1 + \tanh^2 x)$$

Taylor expansion for $x \to 0$

$$\sinh x \simeq x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x \simeq 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$$

$$\tanh x \simeq x - \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$\coth x \simeq \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \ldots$$

Integrals

$$\int \sinh^2(\alpha x)\,dx = \frac{1}{4\alpha}\sinh(2\alpha x) - \frac{x}{2} + C$$

$$\int \cosh^2(\alpha x)\,dx = \frac{1}{4\alpha}\sinh(2\alpha x) + \frac{x}{2} + C$$