Velocity gradient tensor - decomposition and geometrical interpretation

We call $\underline{\underline{\dot{\Pi}}}$ the VELOCITY GRADIENT TENSOR defined as

$$\dot{\Pi}_{ij} \doteq \partial_j v_i \qquad \left(\underline{\underline{\dot{\Pi}}} = \mathrm{grad}(\bar{v}) \right)$$

i.e. a 2nd-order tensor, thus subject to one and only one decomposition. We shall see in the following the interpretation of this tensor and of the components of its decomposition in relation to the motion of a continuum element.

The decomposition is

$$\underline{\underline{\dot{\Pi}}} = \underbrace{\underline{\underline{\dot{\Pi}}}^S}_{\text{traceless symmetric}} + \underbrace{\underline{\underline{\dot{\Pi}}}^A}_{\text{antisymmetric}} + \underbrace{\underline{\underline{\dot{\Pi}}}^I}_{\text{isotropic}}$$

with

$$\underline{\underline{\dot{\Pi}}}^S = \frac{1}{2}\left(\dot{\Pi}_{ij} + \dot{\Pi}_{ji} - \frac{2}{3}\dot{\Pi}_{\ell\ell}\delta_{ij}\right),$$

$$\text{that is } \dot{\Pi}^S_{ij} = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\partial_\ell v_\ell \delta_{ij}\right) = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\mathrm{div}(\bar{v})\delta_{ij}\right)$$

$$\underline{\underline{\dot{\Pi}}}^A = \frac{1}{2}\left(\dot{\Pi}_{ij} - \dot{\Pi}_{ji}\right) = \frac{1}{2}\left(\partial_j v_i - \partial_i v_j\right)$$

$$\underline{\underline{\dot{\Pi}}}^I = \frac{1}{3}\dot{\Pi}_{\ell\ell}\delta_{ij} = \frac{1}{3}\partial_\ell v_\ell \delta_{ij} = \frac{1}{3}\mathrm{div}(\bar{v})\delta_{ij}$$

Now let us consider an element of a continuous medium and specifically two points in it:

O with position $\bar{x}(t)$, velocity $\bar{v}(\bar{x}(t))$,

P with position $\bar{y}(t)$, velocity $\bar{v}(\bar{y}(t))$;

$P-O = \bar{y}-\bar{x}$ is the infinitesimal vector representing the position of P with respect to O;

the velocity of P can be written as

$\bar{v}(\bar{y}) = \bar{v}(\bar{x}) + (\bar{y}-\bar{x})\,\mathrm{grad}(\bar{v}(\bar{x}))$ or, in components,

$v_i(\bar{y}) = v_i(\bar{x}) + (y_j - x_j)\partial_j v_i(\bar{x}) = v_i(\bar{x}) + (y_j - x_j)\dot{\Pi}_{ij} =$ using the decomposition of $\dot{\Pi}_{ij}$

$$= v_i(\bar{x}) + (y_j - x_j)\frac{1}{2}\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\mathrm{div}(\bar{v})\delta_{ij}\right) + (y_j - x_j)\frac{1}{2}\left(\partial_j v_i - \partial_i v_j\right) + (y_j - x_j)\frac{1}{3}\mathrm{div}(\bar{v})\delta_{ij}$$

A careful analysis of all terms will reveal that the motion of the element is a combination of:

1. A <u>translatory (linear) motion</u> (associated to the part of $\bar{v}(\bar{y})$ dictated by $\bar{v}(\bar{x})$);
2. A <u>rotatory motion</u> with angular velocity $\bar{\omega} = \frac{1}{2}\,\mathrm{curl}\,\bar{v}$ (associated to $\underline{\underline{\dot{\Pi}}}^A$);
3. An isotropic expansion/contraction with relative expansion velocity $\frac{1}{V}\frac{dV}{dt} = \mathrm{div}(\bar{v})$ (associated to $\underline{\underline{\dot{\Pi}}}^I$);
4. An anisotropic, isochoric deformation of lengths and angles (associated to $\underline{\underline{\dot{\Pi}}}^S$).

① The term $\bar{v}(\bar{x})$ in $\bar{v}(\bar{y})$ is equal $\forall$ pt $\in$ fluid element and clearly accounts for a rigid displacement of the whole element, where all points do not move with respect to the reference point O ($\Rightarrow \bar{v}(\bar{y}) = \bar{v}(\bar{x})$).

② Let us recall the existence of duality relations between a 2nd-order antisymmetric tensor and a pseudovector; calling $\check{\underline{\Pi}}$ the pseudovector associated to $\dot{\underline{\underline{\Pi}}}^A$, these read

$$\check{\Pi}^i = \frac{1}{2}\varepsilon_{ijk}\dot{\Pi}^A_{jk} \quad ; \quad \dot{\Pi}^A_{ij} = \varepsilon_{ijk}\check{\Pi}_k$$

$$\Rightarrow (y_j - x_j)\frac{1}{2}(\partial_j v_i - \partial_i v_j) = (y_j - x_j)\dot{\Pi}^A_{ij} = (y_j - x_j)\varepsilon_{ijk}\check{\Pi}_k = -\varepsilon_{ikj}\check{\Pi}_k(y_j - x_j) =$$

$$= \left(\text{if we define } \omega_k \doteq -\check{\Pi}_k\right) = \underline{\underline{[\bar{\omega}\times(\bar{y}-\bar{x})]_i}}$$

Notice that $\omega_i = -\check{\Pi}_i = -\frac{1}{2}\varepsilon_{ijk}\dot{\Pi}^A_{jk} = -\frac{1}{4}\varepsilon_{ijk}(\partial_k v_j - \partial_j v_k) =$

$$= -\frac{1}{4}\varepsilon_{ijk}\partial_k v_j + \frac{1}{4}\varepsilon_{ijk}\partial_j v_k = \frac{1}{4}\varepsilon_{ikj}\partial_k v_j + \frac{1}{4}\varepsilon_{ijk}\partial_j v_k = \frac{1}{4}(\text{curl}\,\bar{v})_i + \frac{1}{4}(\text{curl}\,\bar{v})_i =$$

$$= \frac{1}{2}(\text{curl}\,\bar{v}) \quad \Rightarrow \quad \boxed{\bar{\omega} = \frac{1}{2}\text{curl}\,\bar{v}}$$

This term in the relative velocity $\bar{v}(y) - \bar{v}(\bar{x})$ appears to be $= \frac{1}{2}[\bar{\omega}\times(\bar{y}\times\bar{x})]$, which represents a rigid rotation around an axis passing through O, with a rotation angular velocity $\bar{\omega}$.

③ Let us consider the term $(y_j - x_j)\frac{1}{3}\text{div}(\bar{v})\delta_{ij}$ or, in vector notation, $(\bar{y}-\bar{x})\frac{1}{3}\text{div}\,\bar{v}$. For any point N in the continuum element, this term contributes as

$$\bar{v}(N) - \bar{v}(O) = \frac{1}{3}\text{div}(\bar{v}(\bar{x}(O))(\bar{x}(N) - \bar{x}(O))$$

that is a velocity along the N-O direction with a coefficient $\frac{1}{3}\text{div}\,\bar{v}(\bar{x}(O))$ that is equal for all points in the element. If we consider four points in the continuum element O, P_1, P_2, P_3 that define the edges $P_1 - O$, $P_2 - O$, $P_3 - O$ of a parallelepiped of infinitesimal size at the time instant t, a motion of these four points according to this velocity term will yield, at time $t+dt$,

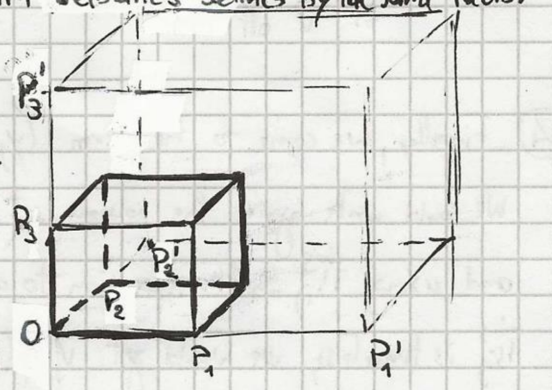
$$\bar{x}' = \bar{x} + \bar{v}(\bar{x})dt \quad \text{for } O' = O(t+dt)$$

$$\bar{y}' = \bar{y} + \bar{v}(\bar{y})dt = \bar{y} + \left[\bar{v}(\bar{x}) + \frac{1}{3}\text{div}(\bar{v}(\bar{x}))(\bar{y}-\bar{x})\right]dt \quad \text{for } P' = P(t+dt) \text{ and similarly for } P_2, P_3$$

$\Downarrow$

finally $\bar{y}' - \bar{x}' = \left[1 + \frac{1}{3}\text{div}(\bar{v}(\bar{x}))\,dt\right](\bar{y}-\bar{x})$

So besides the rigid motion of all points with $\bar{v}(\bar{x})$ (velocity of the reference point O), all points move towards new relative positions with velocities defined by the same factor and along the directions already defined at time t.



This part of the velocity accounts for an isotropic expansion (or contraction), a transformation that preserves the angles.

In this occasion we also recover a result obtained when we evaluated material derivatives of volume integrals and took $F(\bar{x},t)=1$ with its associated integral quantity $\mathcal{F}(t)=\int_{R(t)} F(\bar{x},t)\,d^3x = V$ volume:

$$\frac{DV}{Dt}=\int_{R(t)}\left[\frac{DF}{Dt}+F\,\mathrm{div}\,\bar{v}\right]d^3x = \dots = \int_{R(t)}\mathrm{div}(\bar{v})\,d^3x$$

That is to say that the variation in volume of a continuum element has something to do with the $\mathrm{div}(\bar{v})$. There is yet another way to show it very explicitly, that is let us consider a parallelepiped defined by a vertex O (origin) and three vertices P_1, P_2, P_3 determining the edges $P_i - O$ $(i=1,2,3)$. We can write the volume of the parallelepiped as

$V = \left[\bar{x}(P_1)\times\bar{x}(P_2)\right]\cdot\bar{x}(P_3)$ (where we simply wrote $\bar{x}(P_i)$ instead of a more cumbersome $\bar{x}(P_i)-\bar{x}(O)$ by having $O=$ origin $(0,0,0)$)

or in tensor notation

$V=\varepsilon_{ijk}\,x_j(P_1)\,x_k(P_2)\,x_i(P_3)$

and since $\det\underline{\underline{A}} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$,

if we take $A_{hk}=x_h(P_k)$ we can rewrite the volume as

$V=\varepsilon_{ijk}\,x_j(P_r)\,x_k(P_s)\,x_i(P_t) = \det\left[x_\ell(P_h)\right]$

$$\text{Hence } V'=\det\left[x_i'(P_h)\right]=\det\left[\overbrace{x_i(P_h)}^{\delta_{ij}x_j(P_h)}+\partial_j v_i\,x_j(P_h)\,dt\right]=\det\left[\left(\underline{\underline{\mathbb{1}}}+\dot{\underline{\underline{\Pi}}}\,dt\right)x_j(P_h)\right]=$$
$$=\det\left(\underline{\underline{\mathbb{1}}}+\dot{\underline{\underline{\Pi}}}\,dt\right)\det\left(x_j(P_h)\right)\underset{\text{for dt infinitesimal}}{=}\left(1+\mathrm{Tr}\,\dot{\underline{\underline{\Pi}}}\,dt\right)V$$

Since $\dot{\underline{\underline{\Pi}}}^S$, $\dot{\underline{\underline{\Pi}}}^A$ have zero trace, $\mathrm{Tr}\,\dot{\underline{\underline{\Pi}}} = \mathrm{Tr}\,\dot{\underline{\underline{\Pi}}}^I = \frac{1}{3}\mathrm{div}(\bar{v})\,\delta_{ii} = \mathrm{div}(\bar{v})$

$\Rightarrow V'=\left(1+\mathrm{div}(\bar{v})\,dt\right)V$

$\Rightarrow \quad \dfrac{V'-V}{V} = \mathrm{div}(\bar{v})\,dt$ and $V'-V \approx dt$ infinitesimal variation

$$\Rightarrow \frac{1}{V}\frac{dV}{dt} = \mathrm{div}(\bar{v})$$

(4) Finally, we come to the term $(y_j - x_j)\,\frac{1}{2}\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\,\mathrm{div}\,\bar{v}\,\delta_{ij}\right)$.

We could write again the volume at $t+dt$, $V' = \det\left[x_i'(P_n)\right]$, expanding $x_i(P_n)$ and using $\dot{\Pi}^s_{ij}$ in the expansion to account for this term: But as we have just seen, since $\dot{\Pi}^s_{ij}$ is traceless, we would get $V' = (1 + \mathrm{Tr}\,\dot{\Pi}^s_{ij}\,dt)V = V$, which tells us this part of the motion preserves volume, there is no expansion/contraction (one of the new edges of the original parallelepiped $P_i - O \rightarrow P_i' - O$ must be shorter/longer while the opposite occurs to the other ones).

More specifically, if we work in the principal axis system, where $\dot{\underline{\underline{\Pi}}}^s$ is diagonal

$$\dot{\underline{\underline{\Pi}}}^s = \begin{pmatrix} \alpha_1 & \phi & \phi \\ \phi & \alpha_2 & \phi \\ \phi & \phi & \alpha_3 \end{pmatrix},$$

let us have a parallelepiped built upon three edges $P_i - O$ parallel to the cs axes. The vectors $P_i - O$ must then be eigenvectors of $\dot{\underline{\underline{\Pi}}}^s$; at $t+dt$

$$\bar{x}(P_i') - \bar{x}(O') = (\underline{\underline{1}} + \dot{\underline{\underline{\Pi}}}^s dt)\left[\bar{x}(P_i) - \bar{x}(O)\right] = (1+\alpha_i\,dt)\left[\bar{x}(P_i) - \bar{x}(O)\right]$$

that is to say, the new edges are parallel to the original ones and to the cs axes, so that a right parallelepiped is transformed into a right parallelepiped; as $\sum_{i=1}^{3} \alpha_i = \phi$, one edge is contracted while two are stretched, or viceversa (necessary to preserve volume).

If the parallelepiped does not lie with edges parallel to the principal axes, the components of each edge along the axes are stretched in different amounts, so that each edge changes in orientation while being stretched/contracted; the process is isochoric for small dt, but it does not preserve either lengths or angles: The new parallelepiped is no longer right. The scalar product between two vectors defining two edges is

$$\langle \bar{y}_1' - \bar{x}', \bar{y}_2 - \bar{x}' \rangle = \langle (\underline{\underline{1}} + \dot{\underline{\underline{\Pi}}}^s dt)(\bar{y}_1 - \bar{x}), (\underline{\underline{1}} + \dot{\underline{\underline{\Pi}}}^s dt)(\bar{y}_2 - \bar{x}) \rangle = \langle (\bar{y}_1 - \bar{x}), (\underline{\underline{1}} + \dot{\underline{\underline{\Pi}}}^s dt)^2 (\bar{y}_2 - \bar{x}) \rangle =$$

$$(\text{to 1st-order approx}) = \langle (\bar{y}_1 - \bar{x}), (\underline{\underline{1}} + 2\dot{\underline{\underline{\Pi}}}^s dt)(\bar{y}_2 - \bar{x}) \rangle = \underbrace{\langle (\bar{y}_1 - \bar{x}), (\bar{y}_2 - \bar{x}) \rangle}_{\text{as at time instant } t} + \underbrace{\langle (\bar{y}_1 - \bar{x}), 2\dot{\underline{\underline{\Pi}}}^s dt\,(\bar{y}_2 - \bar{x}) \rangle}_{\text{this term} = \phi \Leftrightarrow \text{edges} \parallel \text{principal axes}}$$

Notes:

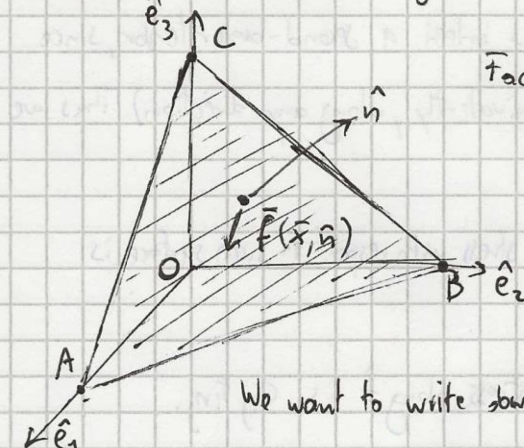
* While $\dot{\underline{\underline{\Pi}}}^{A}$ accounts for a rotation the other two terms yield a deformation, so we can define a STRAIN RATE TENSOR $\dot{\underline{\underline{\varepsilon}}} = \dot{\underline{\underline{\Pi}}}^{S} + \dot{\underline{\underline{\Pi}}}^{I}$ with components $\dot{\underline{\underline{\varepsilon}}}^{S} = \dot{\underline{\underline{\Pi}}}^{S}$, $\dot{\underline{\underline{\varepsilon}}}^{I} = \dot{\underline{\underline{\Pi}}}^{I}$. This symmetric tensor represents the velocity of deformation of a continuum element.

* While a fluid is treated in terms of the velocity gradients and thus $\dot{\underline{\underline{\Pi}}}$, elastic solids are indeed dealt with by working with their deformations or <u>displacements</u>. Conceptually there is no real difference, since a velocity multiplied by dt yields a displacement, and $\dot{\underline{\underline{\Pi}}}\,dt$ therefore yields a deformation gradient tensor. All considerations made so far for $\dot{\underline{\underline{\Pi}}}$ will hold for this tensor, too.

Real (viscous) fluids

Following a procedure already exploited to prove the isotropy of pressure in a fluid either at rest or free from viscosity (ideal), we can find an expression relating surface forces and stresses that is not only valid for any continuum but also an evidence of the fact that stresses constitute a 2nd-order tensor (STRESS TENSOR σ_{ij}) – something we have previously stated without proving it.

Let us consider again a tetrahedral continuum element of infinitesimal diameter ℓ, with vertices O, A, B, C and three faces along the coordinate axes (see figure).



Faces: S_n (ABC), normal unit vector $\hat{n}$, area A_n

S_1 (OBC), normal unit vector $-\hat{e}_1$, area A_1

S_2 (OAC), normal unit vector $-\hat{e}_2$, area A_2

S_3 (OAB), normal unit vector $-\hat{e}_3$, area A_3

We want to write down the resultant of forces. Notice that surface forces are now not only along the normal to each surface as both normal and shear forces are allowed. $\bar{f}(\bar{x},\hat{n})$ will simply indicate the force at position $\bar{x}$ on the face with normal vector $\hat{n}$.

As before summing all forces as if they were evaluated at the same position results in an error of order $O(\ell)$. Surface forces are: $\bar{f}(\bar{x},\hat{n})$ on S_n (stresses considered as usual from the outside towards the face)

$$\bar{f}(\bar{x},-\hat{e}_i) = -\bar{f}(\bar{x},\hat{e}_i) \text{ on } S_i \ (i=1,2,3)$$

We will also have volume forces $\sim \rho V \bar{g} \sim O(\ell^3)$ ($\bar{g}$ = resultant of volume forces per unit mass).

According to Euler's first law of motion, the resultant of forces $\bar{R}$ on the continuum element is equal to the acceleration $\bar{a}$ of the center of mass on the element, times its mass (= rate of change of its linear momentum): $\bar{R} = m\bar{a}$. Recalling that $A_j = A_n(\hat{n}\cdot\hat{e}_j) = A_n n_j$ we can write the i-th component of Euler's first law of motion as ($m = \rho V$)

$$\left[f_i(\bar{x},\hat{n}) - f_i(\bar{x},\hat{e}_1)n_1 - f_i(\bar{x},\hat{e}_2)n_2 - f_i(\bar{x},\hat{e}_3)n_3\right]A_n + \rho V g_i + \underbrace{O(\ell^3)}_{\text{error terms: } O(\ell^2)\text{ stresses}\cdot O(\ell) = O(\ell^3)} - \underbrace{\rho V a_i}_{\sim O(V) = O(\ell^3)} = \phi$$

We require the expression to vanish; since it contains $O(\ell^2)$ and $O(\ell^3)$ terms, these must vanish separately; hence the $[\ldots]A_n$ term ($O(\ell^2)$) vanishes if

$$f_i(\bar{x},\hat{n}) = f_i(\bar{x},\hat{e}_1)n_1 + f_i(\bar{x},\hat{e}_2)n_2 + f_i(\bar{x},\hat{e}_3)n_3$$

and by defining
$$\sigma_{ij}(\bar{x}) = f_i(\bar{x},\hat{e}_j)$$

we get
$$f_i(\bar{x},\hat{n}) = \sigma_{i1}(\bar{x})n_1 + \sigma_{i2}(\bar{x})n_2 + \sigma_{i3}(\bar{x})n_3 = \sigma_{ij}(\bar{x})n_j$$

The expression $\boxed{f_i(\bar{x},\hat{n}) = \sigma_{ij}(\bar{x})n_j}$ is called Cauchy's stress theorem and carries some important consequences.

* $\bar{f}$ and $\hat{n}$ are vectors with arbitrary direction; the "quotient rule" for tensors tells us that, as $f_i = \sigma_{ij}n_j$, σ_{ij} is more than a simple matrix, it is indeed a second-order tensor, since this expression must hold in any coordinate system (or equivalently, along any direction). Thus we can really call $\underline{\sigma}$ STRESS TENSOR.
* For a surface with normal unit vector $\hat{n}$, the normal stress with respect to such surface is $\sigma_{ij}n_in_j$ (projection of the stress along $\hat{n}$); if $\hat{t}$ is a unit vector parallel to the surface, the shear stress along $\hat{t}$ is $\sigma_{ij}t_in_j$.

There is yet another fundamental consideration about $\underline{\sigma}$. In a variety of situations, $\underline{\underline{\sigma}}$ is shown to be, in some cases, symmetric ($\sigma_{ji} = \sigma_{ij}$). This happens for "ordinary" fluids, i.e. non-polar substances (polar ones contain electric or magnetic dipoles, e.g. polymer chains or suspensions of ferromagnetic particles). Bear in mind the fact that some books still come to the conclusion that σ_{ij} is symmetric by applying static principles to dynamics and messing around, or with ambiguous or sketchy, rushed arguments. This result warrants, on the contrary, a thorough discussion we shall present a bit later.

Stress tensor for a Newtonian fluid – Navier-Stokes equation

As we discussed before, the stress tensor σ_{ij} can be decomposed in a pressure term (the only one occurring for ideal fluids) and a remaining part that accounts for non-ideal, i.e. viscous stresses, called σ'_{ij} <u>deviatoric</u> or <u>viscous</u> stress tensor: $\sigma_{ij} = \sigma'_{ij} - p\delta_{ij}$.

The viscous fluid can be called NEWTONIAN, and the corresponding viscous stress tensor σ'_{ij} takes on a special form, when some specific conditions are met:

1. A <u>linear constitutive relation</u> exists between the viscous tensor and the velocity gradient tensor $\dot{\Pi}_{ij}$;
2. The <u>fluid is isotropic</u>;
3. The <u>stress tensor is symmetric.</u>

Condition 1 is equivalent to say that $\sigma'_{ij} = A_{ijhk}\dot{\Pi}_{hk}$.

Condition 2, isotropic fluid, requires that no special cs exist; in other words, the constitutive relation must express the same relationship between stresses and velocity gradients along any observation direction. Therefore the form of the 4th-order tensor A_{ijhk} is of this kind (see Theorem 17, paragraph 2.1 from Segel):

$$A_{ijhk} = a\delta_{ij}\delta_{hk} + b\delta_{ih}\delta_{jk} + c\delta_{ik}\delta_{jh}$$; plugging it into the constitutive relation,

$$\sigma'_{ij} = a\delta_{ij}\delta_{hk}\dot{\Pi}_{hk} + b\delta_{ih}\delta_{jk}\dot{\Pi}_{hk} + c\delta_{ik}\delta_{jh}\dot{\Pi}_{hk} =$$

$$= a\delta_{ij}\dot{\Pi}_{hh} + b\dot{\Pi}_{ij} + c\dot{\Pi}_{ji} =$$ with the decomposition $\dot{\underline{\underline{\Pi}}} = \dot{\underline{\underline{\Pi}}}^S + \dot{\underline{\underline{\Pi}}}^A + \dot{\underline{\underline{\Pi}}}^I$

$$= \underbrace{a(\mathrm{Tr}\,\dot{\Pi}_{ij})\delta_{ij}}_{=3\dot{\Pi}^I_{ij}} + b\dot{\Pi}^S_{ij} + b\dot{\Pi}^A_{ij} + b\dot{\Pi}^I_{ij} + c\underbrace{\dot{\Pi}^S_{ji}}_{=\dot{\Pi}^S_{ij}} + c\underbrace{\dot{\Pi}^A_{ji}}_{=-\dot{\Pi}^A_{ij}} + c\underbrace{\dot{\Pi}^I_{ji}}_{=\dot{\Pi}^I_{ij}} =$$

$$= (b+c)\dot{\Pi}^S_{ij} + (b-c)\dot{\Pi}^A_{ij} + (3a+b+c)\dot{\Pi}^I_{ij}$$

Condition 3 requires σ'_{ij} to be symmetric, hence the coefficient of $\dot{\Pi}^A_{ij}$ must vanish:

$b - c = 0 \Rightarrow \boxed{b = c} \Rightarrow$
$$\sigma'_{ij} = 2b\dot{\Pi}^S_{ij} + (3a+2b)\dot{\Pi}^I_{ij}$$

and calling $\eta \doteq b$, $\zeta \doteq a + \frac{2}{3}b$ first (η) and second (ζ) viscosity coefficients,

$$\boxed{\sigma'_{ij} = 2\eta\dot{\Pi}^S_{ij} + 3\zeta\dot{\Pi}^I_{ij} = 2\eta\dot{\varepsilon}^S_{ij} + 3\zeta\dot{\varepsilon}^I_{ij}}$$

with the explicit expression of $\dot{\Pi}_{ij} = \partial_j v_i$ (and $\dot{\Pi}^S_{ij} = \frac{1}{2}(\dot{\Pi}_{ij} + \dot{\Pi}_{ji} - \frac{2}{3}\mathrm{Tr}(\dot{\Pi}_{ij})\delta_{ij})$, $\dot{\Pi}^I_{ij} = \frac{1}{3}\mathrm{Tr}(\dot{\Pi}_{ij})\delta_{ij}$)

$$\sigma'_{ij} = \eta\left(\dot{u}_{ij} + \dot{u}_{ji} - \frac{2}{3}\operatorname{Tr}(\dot{u}_{ij})\delta_{ij}\right) + \zeta \operatorname{Tr}(\dot{u}_{ij})\delta_{ij} =$$

$$= \underline{\eta\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\operatorname{div}(\bar{v})\delta_{ij}\right) + \zeta \operatorname{div}(\bar{v})\delta_{ij}}$$

$$\Rightarrow \sigma_{ij} = \sigma'_{ij} - p\delta_{ij} = \underline{\eta\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\operatorname{div}(\bar{v})\delta_{ij}\right) + \left(\zeta \operatorname{div}(\bar{v}) - p\right)\delta_{ij}} \quad \text{full stress tensor}$$

If we now reconsider the first law of motion for a fluid element in local (differential) form, considering f force per unit mass, $D\bar{v}/Dt = \bar{f}$;
for an ideal fluid we had $\bar{f} = -\frac{1}{\rho}\operatorname{grad} p = \frac{1}{\rho}\operatorname{grad}(-p)$ where now stresses include p but viscous stresses must be added, so, writing the i-th component,

$$f_i = \frac{1}{\rho}\partial_j \sigma_{ij} + g_i \quad (g_i \text{ i-th component of a volume force } \bar{g});$$

with the expression we obtained for the σ_{ij} of a Newtonian fluid,

$$f_i = \frac{1}{\rho}\partial_j\left[\eta\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\operatorname{div}(\bar{v})\delta_{ij}\right) + \zeta \operatorname{div}(\bar{v})\delta_{ij} - p\delta_{ij}\right] + g_i =$$

$$= \frac{\eta}{\rho}\Big[\underbrace{\partial_j\partial_j v_i}_{\nabla^2 v_i} + \underbrace{\partial_j\partial_i v_j}_{\partial_i\partial_j v_j = \partial_i \operatorname{div}(\bar{v})} - \frac{2}{3}\underbrace{\partial_j \operatorname{div}(\bar{v})\delta_{ij}}_{\partial_i \operatorname{div}(\bar{v})}\Big] + \frac{\zeta}{\rho}\underbrace{\partial_j \operatorname{div}(\bar{v})\delta_{ij}}_{\partial_i \operatorname{div}(\bar{v})} - \frac{1}{\rho}\underbrace{\partial_j p\delta_{ij}}_{\partial_i p} + g_i =$$

$$= \frac{\eta}{\rho}\nabla^2 v_i + \frac{1}{\rho}\left(\frac{1}{3}\eta + \zeta\right)\partial_i(\operatorname{div}\bar{v}) - \frac{1}{\rho}\partial_i p + g_i \quad \text{hence the law of motion}$$

$$\boxed{\frac{D\bar{v}}{Dt} = \frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\operatorname{grad})\bar{v} = -\frac{1}{\rho}\operatorname{grad}p + \frac{\eta}{\rho}\nabla^2\bar{v} + \frac{1}{\rho}\left(\frac{1}{3}\eta + \zeta\right)\operatorname{grad}(\operatorname{div}\bar{v}) + \bar{g}}$$

that is the full Navier-Stokes equation for a viscous Newtonian fluid.

The case we are going to study in the following, i.e. an incompressible flow ($\operatorname{div}(\bar{v}) = 0$) yields a more common form of the Navier-Stokes equation:

$$\boxed{\frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\operatorname{grad})\bar{v} = -\frac{1}{\rho}\operatorname{grad}p + \frac{\eta}{\rho}\nabla^2\bar{v} + \bar{g}}$$

where only the first viscosity coefficient η survives. η is called <u>dynamic viscosity</u> while a <u>kinematic viscosity</u> is defined as $\nu \doteq \eta/\rho$ (notice that ν depends on density, while η does not; imagine for instance a gas whose density $\rho = f(p)$ at a given T; in more general terms, η and ζ are anyway functions of the thermodynamic properties of the continuous medium, e.g., $f(p, T)$ but we often treat them as constants).

Like any partial differential equation, the Navier–Stokes equation must be supplied with proper boundary conditions. While ideal fluids, in the absence of friction, could effortlessly slide along boundary surfaces, hence we only had to set relative normal velocity $= \phi$ (free slip), for a viscous fluid the presence of friction prevents any motion along the solid boundary, so we require

$v_\perp = \phi$ and $v_\parallel = \phi$ relative to the boundary, at any boundary point; overall it is

$\bar{v}_{relative}\Big|_{boundary} = \phi$ (no-slip condition)

At the interface between two immiscible real (viscous) fluids we shall require the same no-slip condition on velocity ($\bar{v}^{(1)} = \bar{v}^{(2)}$), and add a condition on stresses exchanged by the two fluids: stress ① → ② equal and opposite to stress ② → ①, i.e.

$n_k^{(1)} \sigma_{ik}^{(1)} + n_k^{(2)} \sigma_{ik}^{(2)} = \phi$ and since $\hat{n}^{(1)} = -\hat{n}^{(2)} = \hat{n} \Rightarrow n_i \sigma_{ik}^{(1)} = n_i \sigma_{ik}^{(2)}$;

at a free surface (fluid ② is very light ≃ vacuum) no stress, $\Rightarrow \sigma_{ik} n_k = \sigma'_{ik} n_k - p n_i = \phi$

We saw that applying the curl to Euler's equation for an ideal incompressible flow we get

$\frac{\partial}{\partial t}(\text{curl}\,\bar{v}) - \text{curl}(\bar{v} \times \text{curl}\,\bar{v}) = \phi$ (p could be eliminated from the equation);

we can do the same with the Navier–Stokes equation; we get the alternative form

$\frac{\partial}{\partial t}(\text{curl}\,\bar{v}) - \text{curl}(\bar{v} \times \text{curl}\,\bar{v}) = \nu \nabla^2(\text{curl}\,\bar{v})$

and since with $\text{div}\,\bar{v} = \phi$, $\text{curl}(\bar{v} \times \text{curl}\,\bar{v}) = \bar{v}\,\cancel{\text{div}(\text{curl}\,\bar{v})} - \text{curl}\,\bar{v}\,\cancel{\text{div}\,\bar{v}} + (\text{curl}\,\bar{v} \cdot \text{grad})\bar{v} - (\bar{v} \cdot \text{grad})\,\text{curl}\,\bar{v}$

we get $\frac{\partial}{\partial t}\text{curl}\,\bar{v} + (\bar{v} \cdot \text{grad})\,\text{curl}\,\bar{v} - (\text{curl}\,\bar{v} \cdot \text{grad})\bar{v} = \nu \nabla^2 \text{curl}\,\bar{v}$

or $\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \cdot \text{grad})\bar{v} + \nu \nabla^2 \bar{\omega}$ vorticity form of the Navier–Stokes eq.

The Navier–Stokes equation is in general a very complex nonlinear problem, and there is no "general solution" yet, i.e. there is no proof a smooth solution always exists in 3D space for whatever boundary conditions. Therefore the Clay Mathematics Institute has listed it among the "Millennium Prize Problems" in 2000, the seven mathematical problems whose solution is to be awarded with a 1 million US dollar prize. But hey, all this we do for glory, isn't it (if you do physics for the money, you have a serious judgment problem)?