Navier-Stokes equation in non-Cartesian coordinates

The Navier-Stokes equation describing the dynamics of an incompressible viscous flow has a general form

$$\frac{D\bar{v}}{Dt} = \frac{1}{\rho}\,\mathrm{div}(\underline{\underline{\sigma}}) \quad \left(-\mathrm{grad}\,u\right)$$

In Cartesian coordinates, since we knew the form of $\underline{\underline{\sigma}}$ we could cast the N.-S. equation as

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\mathrm{grad})\,\bar{v} = \nu \nabla^2 \bar{v} - \frac{1}{\rho}\,\mathrm{grad}\,p \quad \left(-\mathrm{grad}\,u\right)$$

where the differential operators take on an "easy" form:

$$\mathrm{grad} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \quad ; \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In the following we shall sketch some of the calculations leading to the form of the N.-S. equation in other orthogonal coordinate systems, and in particular for cylindrical and spherical coordinates.

Cylindrical coordinates

1. We shall first calculate the term $(\bar{v}\cdot\mathrm{grad})\bar{v}$ in cylindrical coordinates. The $(\bar{v}\cdot\mathrm{grad})$ operator can be easily expressed as

$$\bar{v}\cdot(\mathrm{grad})_{cyl} = v_r \frac{\partial}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial}{\partial \vartheta} + v_z \frac{\partial}{\partial z}$$

but when we apply it to $\bar{v}$, differentiating the vector $\bar{v}$ also requires differentiation of the cs unit vectors $(\hat{e}_r, \hat{e}_\vartheta, \hat{e}_z)$ as these are not fixed as in cartesian coordinates. Indeed

$$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\vartheta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \phi \quad ; \quad \frac{\partial \hat{e}_z}{\partial \vartheta} = \phi \quad ; \quad \frac{\partial \hat{e}_r}{\partial z} = \frac{\partial \hat{e}_\vartheta}{\partial z} = \frac{\partial \hat{e}_z}{\partial z} = \phi$$

but $\underline{\dfrac{\partial \hat{e}_r}{\partial \vartheta} = \hat{e}_\vartheta}$; $\underline{\dfrac{\partial \hat{e}_\vartheta}{\partial \vartheta} = -\hat{e}_r}$

and therefore

$$v_r \frac{\partial \bar{v}}{\partial r} = v_r\left[\frac{\partial v_r}{\partial r}\hat{e}_r + v_r \cancel{\frac{\partial \hat{e}_r}{\partial r}} + \frac{\partial v_\vartheta}{\partial r}\hat{e}_\vartheta + v_\vartheta \cancel{\frac{\partial \hat{e}_\vartheta}{\partial r}} + \frac{\partial v_z}{\partial r}\hat{e}_z + v_z \cancel{\frac{\partial \hat{e}_z}{\partial r}}\right] =$$

$$= v_r\left[\frac{\partial v_r}{\partial r}\hat{e}_r + \frac{\partial v_\vartheta}{\partial r}\hat{e}_\vartheta + \frac{\partial v_z}{\partial r}\hat{e}_z\right]$$

$$\frac{v_\vartheta}{r}\frac{\partial \bar{v}}{\partial \vartheta} = \frac{v_\vartheta}{r}\left[\frac{\partial v_r}{\partial \vartheta}\hat{e}_r + v_r\frac{\partial \hat{e}_r}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial \vartheta}\hat{e}_\vartheta + v_\vartheta\frac{\partial \hat{e}_\vartheta}{\partial \vartheta} + \frac{\partial v_z}{\partial \vartheta}\hat{e}_z + v_z\cancel{\frac{\partial \hat{e}_z}{\partial \vartheta}}\right] =$$

$$= \frac{v_\vartheta}{r}\left[\frac{\partial v_r}{\partial \vartheta}\hat{e}_r + v_r\hat{e}_\vartheta + \frac{\partial v_\vartheta}{\partial \vartheta}\hat{e}_\vartheta - v_\vartheta\hat{e}_r + \frac{\partial v_z}{\partial \vartheta}\hat{e}_z\right]$$

$$v_z\frac{\partial \bar{v}}{\partial z} = v_z\left[\frac{\partial v_r}{\partial z}\hat{e}_r + v_r\cancel{\frac{\partial \hat{e}_r}{\partial z}} + \frac{\partial v_\vartheta}{\partial z}\hat{e}_\vartheta + v_\vartheta\cancel{\frac{\partial \hat{e}_\vartheta}{\partial z}} + \frac{\partial v_z}{\partial z}\hat{e}_z + v_z\cancel{\frac{\partial \hat{e}_z}{\partial z}}\right] =$$

$$= v_z\left[\frac{\partial v_r}{\partial z}\hat{e}_r + \frac{\partial v_\vartheta}{\partial z}\hat{e}_\vartheta + \frac{\partial v_z}{\partial z}\hat{e}_z\right]$$

We get two sort of "additional" terms, $-\frac{v_\vartheta^2}{r}\hat{e}_r$ e $\frac{v_r v_\vartheta}{r}\hat{e}_\vartheta$.

② In order to express the $\mathrm{div}(\underline{\underline{\sigma}})$ in the equation we shall write down $\underline{\underline{\sigma}}$ (and then the divergence operator, too) in cylindrical coordinates. In general we have

$\sigma_{ij} = \sigma_{ij}' - p\delta_{ij}$ or equivalently $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}' - p\underline{\underline{1}}$

and for an incompressible flow the deviatoric term is reduced to

$\sigma_{ij}' = \eta\left(\partial_j v_i + \partial_i v_j\right)$ or equivalently $\underline{\underline{\sigma}} = \eta\left(\mathrm{grad}\,\bar{v} + (\mathrm{grad}\,\bar{v})^T\right)$

The gradient of $\bar{v}$ can be expressed as $\underline{\bar{\nabla}\bar{v} = (\bar{\nabla}\otimes\bar{v})^T}$, where we have to take (as above) the derivatives of the system's unit vectors: $[\Rightarrow (\bar{\nabla}\bar{v})^T = \bar{\nabla}\otimes\bar{v}]$

$$(\bar{\nabla}\bar{v})^T = \bar{\nabla}\otimes\bar{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\vartheta\frac{1}{r}\frac{\partial}{\partial \vartheta} + \hat{e}_z\frac{\partial}{\partial z}\right)\otimes\left(v_r\hat{e}_r + v_\vartheta\hat{e}_\vartheta + v_z\hat{e}_z\right) =$$

$$= \hat{e}_r\otimes\frac{\partial}{\partial r}(v_r\hat{e}_r) + \hat{e}_r\otimes\frac{\partial}{\partial r}(v_\vartheta\hat{e}_\vartheta) + \hat{e}_r\otimes\frac{\partial}{\partial r}(v_z\hat{e}_z) +$$

$$+ \hat{e}_\vartheta\otimes\frac{1}{r}\frac{\partial}{\partial \vartheta}(v_r\hat{e}_r) + \hat{e}_\vartheta\otimes\frac{1}{r}\frac{\partial}{\partial \vartheta}(v_\vartheta\hat{e}_\vartheta) + \hat{e}_\vartheta\otimes\frac{1}{r}\frac{\partial}{\partial \vartheta}(v_z\hat{e}_z) +$$

$$+ \hat{e}_z\otimes\frac{\partial}{\partial z}(v_r\hat{e}_r) + \hat{e}_z\otimes\frac{\partial}{\partial z}(v_\vartheta\hat{e}_\vartheta) + \hat{e}_z\otimes\frac{\partial}{\partial z}(v_z\hat{e}_z) =$$

$$= \frac{\partial v_r}{\partial r}\hat{e}_r\otimes\hat{e}_r + \frac{\partial v_\vartheta}{\partial r}\hat{e}_r\otimes\hat{e}_\vartheta + \frac{\partial v_z}{\partial r}\hat{e}_r\otimes\hat{e}_z +$$

$$+ \frac{1}{r}\frac{\partial v_r}{\partial \vartheta}\hat{e}_\vartheta\otimes\hat{e}_r + \frac{v_r}{r}\hat{e}_\vartheta\otimes\hat{e}_\vartheta + \frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta}\hat{e}_\vartheta\otimes\hat{e}_\vartheta - \frac{v_\vartheta}{r}\hat{e}_\vartheta\otimes\hat{e}_r + \frac{1}{r}\frac{\partial v_z}{\partial \vartheta}\hat{e}_\vartheta\otimes\hat{e}_z +$$

$$+ \frac{\partial v_r}{\partial z}\hat{e}_z\otimes\hat{e}_r + \frac{\partial v_\vartheta}{\partial z}\hat{e}_z\otimes\hat{e}_\vartheta + \frac{\partial v_z}{\partial z}\hat{e}_z\otimes\hat{e}_z$$

and in matrix form

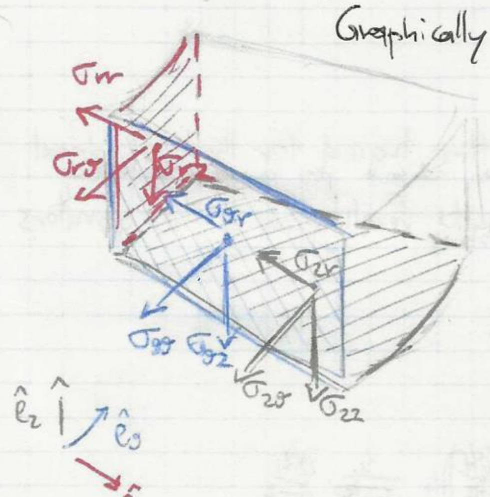
$$(\bar\nabla \bar v)^T = \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\vartheta}{\partial r} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r} & \frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_r}{r} & \frac{1}{r}\frac{\partial v_z}{\partial \vartheta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\vartheta}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

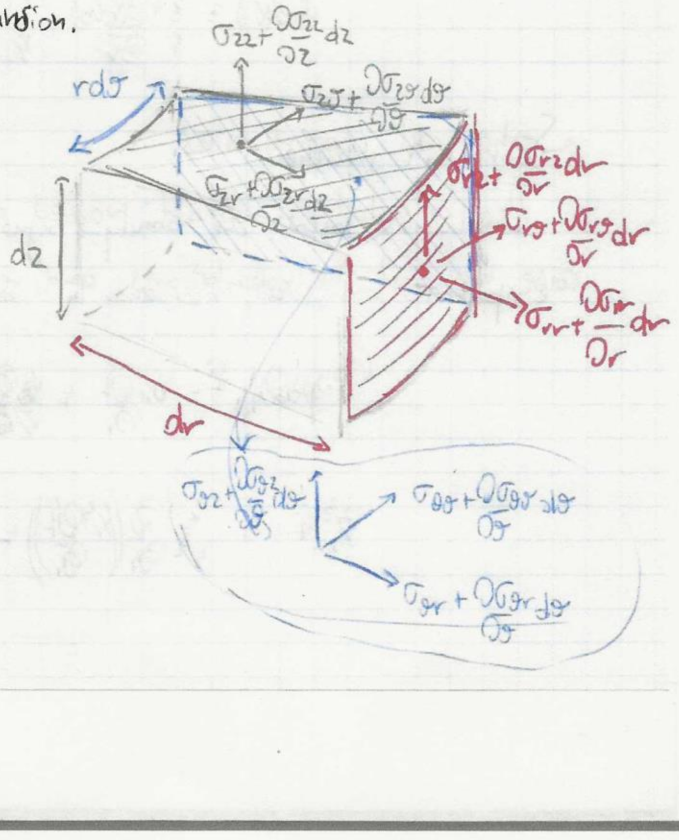
Assembling everything, as $\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{1}}} + \underline{\underline{\sigma}}' = -p\underline{\underline{\mathbb{1}}} + \eta\left(\bar\nabla\bar v + (\bar\nabla \bar v)^T\right)$, we get

$$\boxed{\begin{aligned} \sigma_{rr} &= -p + 2\eta\frac{\partial v_r}{\partial r} & \sigma_{r\vartheta} = \sigma_{\vartheta r} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial r} - \frac{v_\vartheta}{r}\right) \\ \sigma_{\vartheta\vartheta} &= -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_r}{r}\right) & \sigma_{\vartheta z} = \sigma_{z\vartheta} &= \eta\left(\frac{1}{r}\frac{\partial v_z}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial z}\right) \\ \sigma_{zz} &= -p + 2\eta\frac{\partial v_z}{\partial z} & \sigma_{rz} = \sigma_{zr} &= \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \end{aligned}}$$

3 The last step is expressing the div operator in cylindrical coordinates. The procedure is the same as the one applied to obtain the divergence of a vector field, that is, we calculate the flux of the quantity of interest through an infinitesimal volume defined by surfaces normal to the cs' unit vectors $(\hat e_r, \hat e_\vartheta, \hat e_z)$. Hence for each direction we sum all stresses along such direction found on the three pairs of opposite surfaces. Since all lengths are infinitesimal, stresses calculated at an infinitesimal displacement along a certain direction are expressed as a first-order expansion.

Graphically:





We omit here the full calculation; the final result reads

$$\operatorname{div}\underline{\underline{\sigma}} = \left(\frac{\sigma_{rr}}{r} + \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{rz}}{\partial z} - \frac{\sigma_{\vartheta\vartheta}}{r}\right)\hat{e}_r +$$
$$+\left(\frac{2\sigma_{r\vartheta}}{r} + \frac{\partial\sigma_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{\vartheta z}}{\partial z}\right)\hat{e}_\vartheta +$$
$$+\left(\frac{\sigma_{rz}}{r} + \frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\vartheta z}}{\partial\vartheta} + \frac{\partial\sigma_{zz}}{\partial z}\right)\hat{e}_z$$

Summarizing, we can write the Navier–Stokes eq. in its cylindrical components:

(r) $$\frac{\partial v_r}{\partial t} + (\bar{v}\cdot\operatorname{grad})v_r - \frac{v_\vartheta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{2}{r^2}\frac{\partial v_\vartheta}{\partial\vartheta} - \frac{v_r}{r^2}\right)$$

(ϑ) $$\frac{\partial v_\vartheta}{\partial t} + (\bar{v}\cdot\operatorname{grad})v_\vartheta + \frac{v_r v_\vartheta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta} + \nu\left(\nabla^2 v_\vartheta + \frac{2}{r^2}\frac{\partial v_r}{\partial\vartheta} - \frac{v_\vartheta}{r^2}\right)$$

(z) $$\frac{\partial v_z}{\partial t} + (\bar{v}\cdot\operatorname{grad})v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 v_z$$

where we have used the explicit expression of $\underline{\underline{\sigma}}$ in the $\operatorname{div}\underline{\underline{\sigma}}$ term and the laplacian operator in cylindrical coordinates reads

$$\nabla^2_{cyl} f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\vartheta^2} + \frac{\partial^2 f}{\partial z^2}$$

We must add the incompressibility condition

$$\operatorname{div}(\bar{v}) = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta} + \frac{\partial v_z}{\partial z} = 0$$

Spherical coordinates

The calculation should proceed along the outline traced for the cylindrical case. We omit the calculations and report the results, first recalling the operators:

$$(\bar{v}\cdot\operatorname{grad})_{sph} f = v_r\frac{\partial f}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial f}{\partial\vartheta} + \frac{v_\varphi}{r\sin\vartheta}\frac{\partial f}{\partial\varphi}$$

$$\nabla^2_{sph} f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}$$

The components of the stress tensor read

$$
\boxed{
\begin{aligned}
\sigma_{rr} &= -p + 2\eta \frac{\partial v_r}{\partial r} &\qquad \sigma_{r\vartheta} = \sigma_{\vartheta r} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial r} - \frac{v_\vartheta}{r}\right) \\
\sigma_{\vartheta\vartheta} &= -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_r}{r}\right) &\qquad \sigma_{\vartheta\varphi} = \sigma_{\varphi\vartheta} &= \eta\left(\frac{1}{r\sin\vartheta}\frac{\partial v_\vartheta}{\partial \varphi} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \vartheta} - \frac{v_\varphi \cot\vartheta}{r}\right) \\
\sigma_{\varphi\varphi} &= -p + 2\eta\left(\frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\vartheta \cot\vartheta}{r}\right) &\qquad \sigma_{r\varphi} = \sigma_{\varphi r} &= \eta\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)
\end{aligned}
}
$$

The components of the Navier-Stokes equation read

$$
\boxed{
\begin{aligned}
(r)\quad & \frac{\partial v_r}{\partial t} + (\bar{v}\cdot\mathrm{grad})v_r - \frac{v_\vartheta^2 + v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 v_r - \frac{2}{r^2\sin^2\vartheta}\frac{\partial(v_\vartheta \sin\vartheta)}{\partial\vartheta} - \frac{2}{r^2\sin\vartheta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{2v_r}{r^2}\right] \\
(\vartheta)\quad & \frac{\partial v_\vartheta}{\partial t} + (\bar{v}\cdot\mathrm{grad})v_\vartheta + \frac{v_r v_\vartheta}{r} - \frac{v_\varphi^2 \cot\vartheta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta} + \nu\left[\nabla^2 v_\vartheta - \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{2}{r^2}\frac{\partial v_r}{\partial\vartheta} - \frac{v_\vartheta}{r^2\sin^2\vartheta}\right] \\
(\varphi)\quad & \frac{\partial v_\varphi}{\partial t} + (\bar{v}\cdot\mathrm{grad})v_\varphi + \frac{v_r v_\vartheta}{r} + \frac{v_\vartheta v_\varphi \cot\vartheta}{r} = -\frac{1}{\rho r\sin\vartheta}\frac{\partial p}{\partial\varphi} + \nu\left[\nabla^2 v_\varphi + \frac{2}{r^2\sin\vartheta}\frac{\partial v_r}{\partial\varphi} + \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial v_\vartheta}{\partial\varphi} - \frac{v_\varphi}{r^2\sin^2\vartheta}\right]
\end{aligned}
}
$$

and we add once again the incompressibility condition

$$
\mathrm{div}(\bar{v}) = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(v_\vartheta \sin\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial\varphi} = 0
$$