Angular momentum and considerations about the simmetry of $\underline{\underline{\sigma}}$

We discuss some considerations related to the meaning of σ_{ij} being symmetric or not.

① Euler's laws of motion – angular momentum

⊙ According to Euler's first law of motion, the rigid motion of a system is the motion of its center of mass (CM) under the action of the resultant of forces $\bar{R}$, applied to the CM itself. $M\frac{d\bar{v}_{CM}}{dt} = \bar{R}$

⊙ According to Euler's second law of motion, the moment $\bar{T}$ of the resultant of forces is equal to the rate of change of the angular momentum (where we consider an "orbital" angular momentum $\bar{L}$, only, i.e. with a $\bar{r} \times \bar{p}$ structure): $\bar{T} = \dot{\bar{L}} = d\bar{L}/dt$.

⊙ As a consequence of these, one can write

$$\frac{d\bar{L}_{CM}}{dt} = \frac{d}{dt}\left(M\bar{x}_{CM} \times \bar{v}_{CM}\right) = \bar{x}_{CM} \times \bar{R} \quad \text{moment of forces applied to the CM}$$

If we consider a continuum element of position $\bar{x}$ and velocity $\bar{v}$ we can easily verify that the angular momentum per unit mass $\bar{\ell} = \lim_{V \to \phi} \bar{L}/M$ coincides with the angular momentum of the CM $\bar{\ell}_{CM}$ by eliminating higher-order infinitesimal terms:

$$\ell_i = \lim_{V \to \phi} \frac{1}{M} \int_R \rho\, \varepsilon_{ijn} x_j v_n\, d^3x = \qquad \text{(omiting the limit; } M, V \text{ mass and volume of the element filling the region } R\text{)}$$

$$= \frac{1}{M}\int_R \rho\, \varepsilon_{ijn}\left[(x_j - x_{CM_j}) + x_{CM_j}\right](v_n - v_{CM_n})d^3x + \frac{1}{M}\int_R \rho\, \varepsilon_{ijn}\left[(x_j - x_{CM_j}) + x_{CM_j}\right]v_{CM_n}d^3x =$$

$$\underbrace{O(\Delta x)}_{R} \qquad \underbrace{\Delta v \sim \frac{dv}{dx}(\Delta x) \sim O(\Delta x)}$$

$$= \frac{1}{M}\int_R \rho\, \varepsilon_{ijn}\underbrace{(x_j - x_{CM_j})(v_n - v_{CM_n})}\,d^3x + \longrightarrow o((\Delta x)^3) \qquad \Leftarrow O((\bar{x} - \bar{x}_{CM})^5)$$

$$+ \frac{1}{M}\int_R \rho\, \varepsilon_{ijn} x_{CM_j}(v_n - v_{CM_n})\,d^3x + \qquad\qquad \leftarrow O((\bar{x} - \bar{x}_{CM})^4) \Bigg\} \text{ negligible}$$

$$+ \frac{1}{M}\int_R \rho\, \varepsilon_{ijn}(x_j - x_{CM_j})v_n\,d^3x + \qquad\qquad \leftarrow O((\bar{x} - \bar{x}_{CM})^4)$$

$$+ \frac{1}{M}\int_R \rho\, \varepsilon_{ijn} x_{CM_j}v_{CM_n}\,d^3x \simeq \begin{array}{l}\text{by approximation}\\ \text{of the integral with the integrand}\end{array} \qquad \leftarrow O((\bar{x} - \bar{x}_{CM})^3) \text{ leading term}$$

$$\simeq \frac{1}{M}\rho\, \varepsilon_{ijn} x_{CM_j} v_{CM_n} V = \varepsilon_{ijn}\bar{x}_{CM_j}\bar{v}_{CM_n} = \ell_{CM_i} \qquad \Rightarrow \quad \bar{\ell} = \bar{\ell}_{CM}$$

② Moment of external forces

We want to write down the resultant of the moment of external forces per unit mass, that is

$\bar{\tau} = \lim\limits_{V \to \phi} \bar{T}/M$ for a continuum region R with mass M and volume V. We shall consider all possible contributions, i.e. the stress tensor $\underline{\underline{\sigma}}$ ($=$ surface forces), volume forces $\bar{f}$ and volume force couples $\bar{c}$ (force couples, or pure moments, are systems of forces that affect a system causing pure rotation without translation; examples include electric/magnetic fields acting on electric/magnetic dipoles embedded in the continuum, hence proportional to the density of dipoles and to the volume of the continuum element). In components,

$$\tau_i = \lim\limits_{V \to \phi} \frac{1}{M} \int\limits_{\partial R} \varepsilon_{ijk} x_j \underbrace{\sigma_{he} n_e}_{\text{surface force}} da + \frac{1}{M} \int\limits_R \varepsilon_{ijk} x_j \underbrace{\rho f_k d^3x}_{\text{volume force}} + \frac{1}{M} \int\limits_R \underbrace{\rho c_i d^3x}_{\text{force couple}} =$$

$=$ omitting the limit and using the divergence theorem

$$= \frac{1}{M} \int\limits_R \varepsilon_{ijk} \partial_e (x_j \sigma_{he}) d^3x + \frac{1}{M} \int\limits_R \rho \varepsilon_{ijk} x_j f_k d^3x + \frac{1}{M} \int\limits_R \rho c_i d^3x =$$

$$= \frac{1}{M} \int\limits_R \varepsilon_{ijk} \Big[\underbrace{(\partial_e x_j)}_{\delta_{je}} \sigma_{he} + x_j \partial_e \sigma_{he} \Big] d^3x + \frac{1}{M} \int\limits_R \rho \varepsilon_{ijk} x_j f_k d^3x + \frac{1}{M} \int\limits_R \rho c_i d^3x =$$

$$= \frac{1}{M} \int\limits_R \varepsilon_{ijk} (\sigma_{hj} + x_j \partial_e \sigma_{he}) d^3x + \text{idem} + \text{idem} = \quad \text{by approximation of the integral over a small region } R$$

$$\simeq \frac{1}{M} \varepsilon_{ijk} (\sigma_{hj} + x_j \partial_e \sigma_{he}) V + \frac{1}{M} \rho \varepsilon_{ijk} x_j f_k V + \frac{1}{M} \rho c_i V \quad \Rightarrow$$

$$\boxed{\tau_i = \frac{1}{\rho} \varepsilon_{ijk} x_j (\partial_e \sigma_{he} + \rho f_k) + \frac{1}{\rho} \varepsilon_{ijk} \sigma_{hj} + c_i}$$

③ Orbital and spin angular momentum

So far we have considered angular momentum of "orbital" type, i.e. in a $\bar{r} \times \bar{v}$ form. Let us assume the total angular momentum per unit mass $\bar{g}$ to be the sum of such a term $\bar{\ell}$ and a second component $\bar{s}$ which we shall interpret later: $\bar{g} = \bar{\ell} + \bar{s}$. $\qquad \bar{\tau} = \overline{(\bar{\ell} + \bar{s})}$

Euler's second law of motion states that

$$\bar{\tau} = \dot{\bar{g}} = \dot{\bar{\ell}} + \dot{\bar{s}} = \dot{\bar{\ell}}_{cH} + \dot{\bar{s}}_{cH} \qquad \text{or} \qquad \tau_i = \dot{\ell}_{cHi} + \dot{s}_{cHi}$$

hence using the expression obtained above for τ_i;

$$\tau_i = \frac{1}{\rho} \varepsilon_{ijk} x_j (\partial_e \sigma_{he} + \rho f_k) + \frac{1}{\rho} \varepsilon_{ijk} \sigma_{hj} + c_i = \dot{\ell}_{cHi} + \dot{s}_{cHi}$$

and since $\bar{\ell}$ has a structure $\bar{r} \times \bar{v}$ (in tensor notation, $\varepsilon_{ijk} x_j v_k$) we can make the association

$$\dot{\ell}_i = \dot{\ell}_{cm_i} = \frac{1}{\rho}\, \varepsilon_{ijh}\, x_j (\partial_e \sigma_{he} + \rho\, f_h)$$

so that it must be $\qquad s_i = \dot{s}_{cm_i} = \frac{1}{\rho}\, \varepsilon_{ijh}\, \sigma_{hj} + c_i$

i.e. this angular momentum term has a different structure (not in the form $\bar{x} \times \bar{v}$), and specifically it is not dependent on the choice of any particular pivot. Hence it is a type of (classical) "spin".

Furthermore, we can show that the term $\frac{1}{\rho}\, \varepsilon_{ijh}\, \sigma_{hj}$ is not zero only if $\underline{\underline{\sigma}}$ has an antisymmetric component; indeed if we explicitly calculate $\varepsilon_{ijh}\, \sigma_{hj}$,

$$\text{for} \quad i=1 \quad \varepsilon_{ijh}\, \sigma_{hj} = \sigma_{32} - \sigma_{23}$$
$$\text{for} \quad i=2 \quad \varepsilon_{2jh}\, \sigma_{hj} = \sigma_{31} - \sigma_{13}$$
$$\text{for} \quad i=3 \quad \sigma_{3jh}\, \sigma_{hj} = \sigma_{12} - \sigma_{21}$$

We can conclude that the spin-like angular momentum $\bar{s}$ has an influence on the system's dynamics (by having $\bar{s} \neq \emptyset \Rightarrow$ generating a contribution in the moment of forces $\bar{\tau}$) only if either $\underline{\underline{\sigma}}$ has an antisymmetric part or volume force couples $\bar{c}$ occur.

④ Conclusions

⊛ STATICS : Equilibrium requires → $\bar{R} = \emptyset$ (vanishing resultant of forces)
↘ $\bar{\tau} = \emptyset$ (vanishing moment of resultant of forces $\bar{R}$)

The moment $\bar{\tau}$ must be zero regardless of the pivot point ; $\Rightarrow$ a system is equilibrium if

⊙ When external force couples $\bar{c} = \emptyset$, $\Rightarrow \frac{1}{\rho}\, \varepsilon_{ijh}\, \sigma_{hj} = \emptyset \quad \Rightarrow \sigma_{ij}^{A} = \emptyset$ ($\underline{\underline{\sigma}}$ is symmetric).

⊙ When $\bar{c} \neq \emptyset \Rightarrow \frac{1}{\rho}\, \varepsilon_{ijh}\, \sigma_{hj} + c_i = \emptyset \Rightarrow \underline{\underline{\sigma}}$ cannot be symmetric (indeed it is not e.g. in polar liquids where $\bar{c} \neq \emptyset$).

⊛ DYNAMICS : From the considerations made so far, the observed symmetry of $\underline{\underline{\sigma}}$ in "common" Newtonian, non-polar, non-ferromagnetic fluids is equivalent to saying that

⊙ There is no spin term in the angular momentum;

⊙ The continuum is not subject to volume force couples (body couples or body moments).

Note: Some books erroneously "prove" that $\underline{\underline{\sigma}}$ is symmetric in the dynamic case simply demonstrating it for the static case, and extending $\bar{\tau} = \dot{\bar{L}} = \emptyset$ straight away. While the conclusion is fine, the argument should be discussed with more serious consideration.

Pressure in a fluid in motion — Mechanical and thermodynamic pressure, local equilibrium

Pressure was found to be isotropic for ideal fluids or fluids at rest; now we want evaluate the normal stresses in the more general case of a viscous fluid, with a full stress tensor $\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$.

The normal stress on a surface with unit normal vector $\hat{n}$ at position $\bar{x}$ is $\quad [n_i n_j \delta_{ij} = n_i n_i = 1]$

$$\sigma_n(\bar{x}) = f_i(\bar{x}) n_i = \sigma_{ij}(\bar{x}) n_j n_i = -p\underbrace{n_j n_i \delta_{ij}}_{=1} + \eta(\partial_j v_i + \partial_i v_j) n_i n_j - \frac{2}{3}\eta \, div(\bar{v}) \underbrace{n_i n_j \delta_{ij}}_{i=j} + \zeta(div(\bar{v})) \underbrace{n_i n_j \delta_{ij}}_{=1}$$

where the term $(\partial_j v_i + \partial_i v_j)$ is not necessarily isotropic, that is, it depends on the orientation, unlike the other parts of σ_n. We can perform an average of the normal stress σ_n over all directions and call it MECHANICAL PRESSURE p_{mech}:

$$p_{mech}(\bar{x}) \doteq -\langle \sigma_{ij} n_i n_j \rangle = -\sigma_{ij}\langle n_i n_j \rangle = -\sigma_{ij}\frac{1}{3}\delta_{ij} = -\frac{1}{3}\sigma_{ii}$$

σ_{ij} can be brought out of the average as it does not depend on $\hat{n}$

Hence in general, for any fluid in motion (including non-Newtonian fluids),

$$p_{mech} = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}(p\delta_{ii} + \sigma'_{ii}) = p - \frac{1}{3}\sigma'_{ii}$$

where it is apparent that p_{mech} is not equal to the pressure as we meant it before (thermodynamic pressure — a quite sophisticated concept we shall reconsider soon).

If we express explicitly σ_{ij} for a Newtonian fluid,

$$p_{mech}(\bar{x}) = -\frac{1}{3}\sigma_{ij}\delta_{ij} = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}\left[\eta\underbrace{(\partial_i v_i + \partial_i v_i - \frac{2}{3}div(\bar{v})\delta_{ii})}_{=div(\bar{v})} + \underbrace{\zeta \, div(\bar{v})\delta_{ii}}_{=3} - p\delta_{ii}\right] =$$

$$= p - \zeta \, div(\bar{v}) \qquad \Rightarrow \quad \boxed{p_{mech} = p - \zeta \, div(\bar{v})}$$

where we can see that indeed, when the fluid is at rest ($\bar{v}=\phi$), we recover simply p. Interestingly, this also happens for incompressible flows, as $div(\bar{v})=\phi$.

We should reconsider the meaning of "p", as it is a thermodynamic quantity and these are, at least in general, something that is defined at equilibrium: which is not the case, for if the fluid is in motion, it will not be at mechanical (let alone thermodynamic) equilibrium. We can circumvent this if we can assume LOCAL EQUILIBRIUM, which is acceptable if the fluid is <u>not too far</u> from global equilibrium:

⊛ The fluid domain is decomposed into elements small enough to be considered each in equilibrium within itself, so that we can define at least locally the thermodynamic quantities and their differential expressions that describe the equations of state:

⊛ Variables that are not rigorously defined can be formulated using the equations of state based upon the state variables previously defined at a local level;

⊛ Finally, integration of the differential forms written at a local level will "put together" the quantities calculated locally over the various fluid elements that are not in equilibrium; the absence of a fully global equilibrium in turn, is apparent from the spatial gradients of the thermodynamic variables over the domain (equations of state hold only in a differential form, i.e. locally, since they are defined at equilibrium and equilibrium is only local).

In order to build all thermodynamic quantities we need at least two; indeed we find out we can always define two.

① Mass density $\rho(\bar{x}) = 1/v(\bar{x})$: This is always a thermodynamic state variable; it is <u>always defined</u>, even locally, and <u>measurable</u>.

② Internal energy per unit mass ε : We cannot measure it directly, but we can still define it reasoning at the microscopic level.

We consider a fluid element as a collection of microscopic constituents; the mechanical energy of those is

$$\bar{E}_T = \sum_r \frac{1}{2} m_r v_r^2 + \sum_r I_r \omega_r^2 + \sum_r U_r + \sum_{r,s} U_{rs} =$$

$$= \underbrace{(\bar{E}_{KCM})}_{} + \underbrace{(\bar{E}_{Krel})}_{} + \underbrace{(U_{EXT})}_{} + \underbrace{(U_{INT})}_{} = \text{potential energy from interaction between constituent particles (or internal degrees of freedom)}$$

kinetic energy of the CM | Kinetic energy of the relative motion of particles with respect to the CM | potential energy from external fields

$$= \underbrace{(\bar{E}_{KCM} + U_{EXT})}_{} \qquad + \qquad (\bar{E}_{Krel} + U_{INT})$$

kinetic energy + potential energy from external fields | internal energy interpreted as energy of internal degrees of freedom : This is indeed the internal energy ε from thermodynamics

Once we have defined ρ, ε thermodynamic state variables, we can build all other thermodynamic quantities by using locally the equations of state and especially we can write $p(\bar{x})$, thermodynamic pressure that enters p_{mech}. Notice that the measurement and the calculation of a thermodynamic quantity may differ in a state that is out of equilibrium (state equations hold only at equilibrium).

Momentum flux — Momentum flux density tensor

As we did for ideal fluids, we can write down the momentum flux density $\overline{\Pi}_{ij}$ — whose form shall be similar to the ideal fluid counterpart, which read $\overline{\Pi}_{ij} = \rho v_i v_j + p \delta_{ij}$. Here there must be something including the whole stress tensor σ_{ij} instead of pure pressure.

From the general form of the equation of motion

$$\rho \frac{D v_i}{dt} = \partial_j \sigma_{ij} + \rho \partial_i u$$

and since $\rho \frac{D v_i}{Dt} = \frac{\partial}{\partial t}(\rho v_i) + \text{div}(\rho v_i \vec{v}) = \frac{\partial}{\partial t}(\rho v_i) + \partial_j (\rho v_i v_j)$

we can equate $\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{D v_i}{Dt} - \partial_j(\rho v_i v_j) = \partial_j \sigma_{ij} + \rho \partial_i u - \partial_j(\rho v_i v_j)$

$$\Rightarrow \frac{\partial}{\partial t}(\rho v_i) = \partial_j(\sigma_{ij} - \rho v_i v_j) + \rho \partial_i u$$

and by integration over a region R that is fixed in time,

$$\int_R \frac{\partial}{\partial t}(\rho v_i)\, d^3x = \frac{d}{dt}\int \rho v_i\, d^3x = \frac{dP_i}{dt} = \int_R \partial_j(\sigma_{ij} - \rho v_i v_j)\, d^3x + \int_R \rho \partial_i u\, d^3x \text{ and with the divergence theorem}$$

$$-\frac{dP_i}{dt} = \int_{\partial R}(\rho v_i v_j - \sigma_{ij}) n_j\, da - \rho \int_R \partial_i u\, d^3x$$

that is to say, the decrease of $\vec{P}$ linear momentum within R is due to a term accounting for external volume force fields and to a flux term, a flux of momentum through the boundary ∂R of the region R; the integrand must be a momentum flux density.

$$\boxed{\Pi_{ij} = \rho v_i v_j - \sigma_{ij}}$$

(indeed we recover the form we already knew for ideal fluids where $\sigma_{ij} = -p\delta_{ij}$)

Dissipation in a viscous fluid

The power per unit mass over an infinitesimal surface da supplied by contact (surface) forces can be written as

$$\frac{1}{m}\,\bar{v}\cdot d\bar{F} = \frac{1}{m}\,v_i(\sigma_{ij} n_j\, da)$$

and integrating over the whole surface ∂R enclosing a region R of fluid,

$$P_S = \int_{\partial R} \frac{1}{m}\, v_i \sigma_{ij} n_j\, da = \int_R \frac{1}{m}\, \partial_j(v_i \sigma_{ij})\, d^3x \simeq \frac{1}{\rho}\,\partial_j(v_i \sigma_{ij}) \qquad \text{using the divergence theorem and approximating the integral over } \partial\text{esimal } R$$

An external volume force potential u will supply a power per unit mass

$$P' = -\frac{Du}{Dt} = -\frac{\partial u}{\partial t} - \bar{v}\cdot\text{grad}\,u$$

$$u = \phi \text{ for time-independent potential}$$

Therefore the energy balance per unit mass is

$$(*)\quad \frac{D}{Dt}\left(\frac{1}{2}v^2 + \varepsilon\right) = \frac{1}{\rho}\,\partial_j(v_i\sigma_{ij}) - \frac{Du}{Dt} \quad\rightsquigarrow\quad \frac{D}{Dt}\left(\frac{1}{2}v^2 + \varepsilon + u\right) = \frac{1}{\rho}\,\partial_j(v_i\sigma_{ij})$$

The rate of change of mechanical energy only is

$$(**)\quad \frac{D}{Dt}\left(\frac{1}{2}v^2 + u\right) = \frac{D}{Dt}\left(\frac{1}{2}v^2\right) + \frac{Du}{Dt} = \bar{v}\cdot\frac{D\bar{v}}{Dt} + \frac{Du}{Dt} = \bar{v}\cdot\frac{D\bar{v}}{Dt} + \bar{v}\cdot\text{grad}\,u \qquad \left(= v_i\frac{Dv_i}{Dt} + v_i\,\partial_i u\right)$$

But $\dfrac{D\bar{v}}{Dt} = \bar{f}$ force per unit mass $= \dfrac{1}{\rho}\,\text{div}\,\underline{\sigma} - \text{grad}\,u \qquad \left(f_i = \dfrac{1}{\rho}\,\partial_j\sigma_{ij} - \partial_i u\right)$

$$\Rightarrow \bar{v}\cdot\frac{D\bar{v}}{Dt} = \bar{v}\cdot\bar{f} = \bar{v}\cdot\frac{1}{\rho}\,\text{div}\,\underline{\sigma} - \bar{v}\cdot\text{grad}\,u \qquad \left(v_i\frac{Dv_i}{Dt} = \frac{1}{\rho}\,v_i\,\partial_j\sigma_{ij} - v_i\,\partial_i u\right)$$

which inserted into $(**)$ yields $\dfrac{D}{Dt}\left(\dfrac{1}{2}v^2 + u\right) = \dfrac{1}{\rho}\,v_i\,\partial_j\sigma_{ij} = \underbrace{\dfrac{1}{\rho}\,\partial_j(v_i\sigma_{ij})}_{P_S =} - \underbrace{\dfrac{1}{\rho}\,\sigma_{ij}\,\partial_j v_i}_{}$

comparing $(*)$ and $(**)$ $\quad\dfrac{D\varepsilon}{Dt} =$

The variation in mechanical energy is due to power supplied by surface forces and to a term that is a dissipation of kinetic energy (turned into ε); this is evident at least for incompressible flow: $\dfrac{1}{\rho}\underbrace{\sigma'_{ij}}_{\bar{u}_{ij}}\partial_j v_i = \dfrac{1}{\rho}\left(2\eta\,\bar{u}^s_{ij} + 3\zeta\,\overline{u}^2_{ij}\right)\left(\bar{u}^s_{ij} + \overline{u}^2_{ij}\right) = \dfrac{1}{\rho}\,2\eta\left(\bar{u}^s_{ij}\right)^2$

$$\overline{u}^2_{ij} = \frac{1}{3}\,\text{div}(\bar{u})\delta_{ij} = \phi \text{ for incompressible flow}$$

$$\Rightarrow \frac{1}{2}\sigma_{ij}\partial_j v_i = \frac{2\eta}{\rho}\left[\frac{1}{2}(\partial_j v_i + \partial_i v_j)\right]^2 = \frac{\nu}{2}\left(\partial_j v_i + \partial_i v_j\right)^2 > \phi \text{ as it is a squared quantity;}$$

in turn, one could say that since dissipation is a positive quantity, viscosity coefficients must be positive. $\dfrac{\nu}{2}\left(\partial_j v_i + \partial_i v_j\right)^2$ is the power dissipated by viscous effects per unit mass.

$\left[\,* = \text{We omit the part } -p\,\delta_{ij} \text{ of } \sigma_{ij} \text{ since the product yields } -p\,\bar{u}^s_{ij}\delta_{ij} = \phi \text{ as } \bar{u}^s_{ii} = \phi\,\right]$