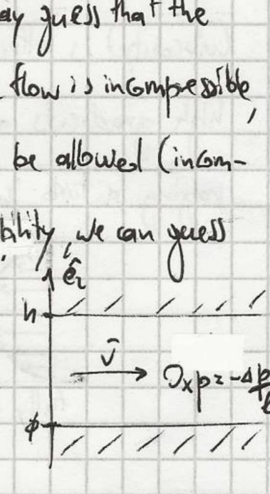
Poiseuille flow

Poiseuille flow is a typical example of viscous, incompressible flow and refers to a <u>pressure-driven</u> flow between static boundaries, i.e. where the fluid is set in motion by a pressure gradient.

Plane geometry

We assume an incompressible flow between two plane, unbounded parallel and static boundaries parallel to a horizontal plane, say $z=\phi$ and $z=h$. These solid boundaries are static (hence no drag on the fluid). We apply a pressure such that there is a uniform gradient in $\hat{e}_x$: $\underline{\partial p/\partial x = -\Delta p/\ell}$ ($\Rightarrow \ell$ shall be a characteristic length for the pressure drop). We look for a steady-state solution.

The system is y-invariant and also x-invariant (as $\partial p/\partial x$ is uniform); we may guess$^{(*)}$ that the solution should feature the same invariances, so we look for $\bar{v} = \bar{v}(z)$; as the flow is incompressible, $\mathrm{div}(\bar{v}) = \phi \Rightarrow \cancel{\partial_x v_x(z)} + \cancel{\partial_y v_y(z)} + \partial_z v_z(z) = \phi$ so only a uniform v_z would be allowed (incompatible with solid boundaries $\perp v_z$); coherently with the invariances and incompressibility, we can guess $\underline{\bar{v} = v_x(z)\hat{e}_x}$, which shall be plugged into the general problem



$$\begin{cases} \dfrac{D\bar{v}}{Dt} = \nu \bar{\nabla}^2 \bar{v} - \dfrac{1}{\rho}\,\mathrm{grad}\,p + \bar{g} \quad (\bar{g} = -g\hat{e}_z \text{ vertical gravity}) \; ; \; \mathrm{grad}\,p = -\dfrac{\Delta p}{\ell}\hat{e}_x \\ \left.\begin{array}{l} v_x(z=\phi) = \phi \\ v_x(z=h) = \phi \end{array}\right\} \text{no-slip condition at boundaries } (v_z = \phi \text{ by the choice of the guess solution}) \\ \mathrm{div}\,\bar{v} = \phi \quad (\text{satisfied by the choice } \bar{v} = v_x(z)\hat{e}_x) \end{cases}$$

The component of the Navier-Stokes equation along $\hat{e}_x$ reads

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + \left(\cancel{v_x \frac{\partial}{\partial x}} + \cancel{v_y \frac{\partial}{\partial y}} + \cancel{v_z \frac{\partial}{\partial z}}\right) v_x = \phi = \nu \frac{d^2 v_x}{dz^2} - \frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow \frac{d^2 v_x}{dz^2} = -\frac{1}{\eta}\frac{\Delta p}{\ell}$$

which, by integration, yields $\dfrac{dv_x}{dz} = -\dfrac{\Delta p}{\eta\ell} z + C_1 \leadsto \underline{v_x(z) = -\dfrac{\Delta p}{2\eta\ell} z^2 + C_1 z + C_2}$

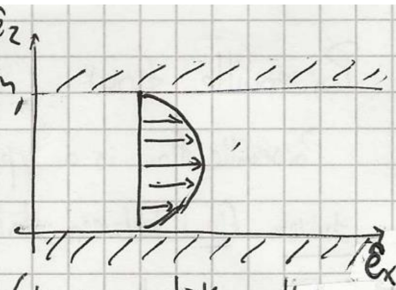
with the b.c. $v_x(\phi) = C_2 = \phi \Rightarrow \boxed{C_2 = \phi}$; $v_x(h) = -\dfrac{\Delta p}{2\eta\ell} h^2 + C_1 h = \phi \Rightarrow \boxed{C_1 = \dfrac{\Delta p}{2\eta\ell} h}$

$$\Rightarrow v_x(z) = \frac{\Delta p}{2\eta\ell}(hz - z^2) \qquad \left(= \frac{\Delta p\, h^2}{2\eta\ell}(\tilde{z} - \tilde{z}^2) \text{ with } \tilde{z} \doteq z/h \text{ normalized height} \right)$$

1) The velocity v_x has a parabolic profile; adjacent velocity vectors are parallel and differ

(*) = This is a choice we make, the only one yielding an analytical solution. Yet there are infinite solution of turbulent nature, $\Rightarrow$ neither respecting the system invariances point by point and at any time instant, nor steady solutions (only averaging over time makes these solution appear "statistically regular").

by infinitesimal increments, so this is regular (LAMINAR) motion, yet notice that



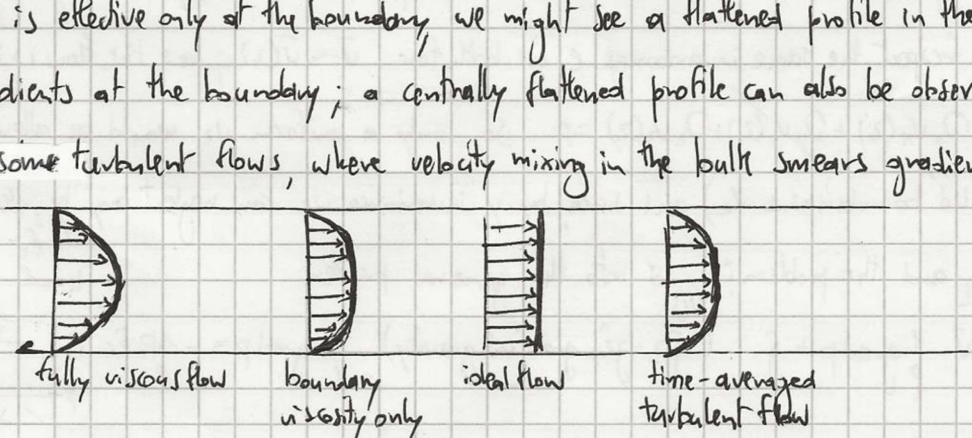
$$\operatorname{curl} \vec v = (\cancel{\partial_y v_z} - \cancel{\partial_z v_y})\hat e_x + (\partial_z v_x - \cancel{\partial_x v_z})\hat e_y + (\cancel{\partial_x v_y} - \cancel{\partial_y v_x})\hat e_z =$$

$$= \partial_z v_x = \frac{\Delta p}{2\eta l}(h-2z) \neq 0$$ this is a rotational flow (leave a stick on the surface of a stream: while it is carried away, it rotates).

⊙ We could calculate the average velocity (and hence the mass flow rate):

$$\tilde v_x = \frac{1}{h}\int_0^h v_x(z)\,dz = \frac{1}{h}\frac{\Delta p}{2\eta l}\left[\frac{hz^2}{2} - \frac{z^3}{3}\right]_0^h = \frac{\Delta p h^2}{12\eta l}$$

⊙ An ideal flow would feature a uniform velocity from boundary to boundary; where friction (viscosity) is effective only at the boundary, we might see a flattened profile in the middle and high gradients at the boundary; a centrally flattened profile can also be observed by averaging some turbulent flows, where velocity mixing in the bulk smears gradients.



⊙ We can calculate explicitly the stress tensor $\sigma_{ij} = \eta(\partial_j v_i + \partial_i v_j - p\delta_{ij})$:

$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$; $\sigma_{xy} = \sigma_{yz} = 0$;

$\sigma_{xz} = \sigma_{zx} = \eta\,\partial_z v_x = \frac{\Delta p}{2\eta}(h-2z) = \beta$

$$\Rightarrow \underline{\underline{\sigma}} = \begin{pmatrix} -p & 0 & \beta \\ 0 & -p & 0 \\ \beta & 0 & -p \end{pmatrix}$$

and we can calculate the shear stress (friction) on the boundaries; on the bottom one, we have σ_{zx} (stress along x on a surface with outward normal vector along z):

$$\sigma_{zx} = \eta\frac{\partial v_x}{\partial z}\Big|_{z=0} = h\Delta p/2l \; ; \text{ on the top one, } (-)\sigma_{zx} = -\eta\frac{\partial v_x}{\partial z}\Big|_{z=h} = h\Delta p/2l$$

the normal vector is $-\hat e_z$

(both walls are "pushed" forward, and apply equal and opposite resistance along $-\hat e_x$ to the fluid).

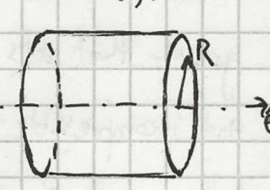
⊙ The z-component of the Navier-Stokes equation reads

$$\frac{Dv_z}{Dt} = 0 = \nu\nabla^2 v_z - \frac{1}{\rho}\frac{\partial p}{\partial z} - g \Rightarrow \frac{1}{\rho}\partial_z p = -g$$ hydrostatic equilibrium equation

(we ignore the y-component as negligible by y-invariance; basically, it is a 2D problem).

Cylindrical geometry - Flow in a pipe

We consider here a steady, viscous and incompressible flow along a pipe of uniform radius R, caused by a longitudinal pressure gradient $\frac{\partial p}{\partial z} = -\frac{\Delta p}{\ell}$ (uniform along the axis $\hat{e}_z$). When guessing the form of the solution, we exploit similar considerations to those used for the planar case; here we notice a rotational invariance as well as a translational invariance along z, and we guess a solution



$\bar{v} = v_z(r)\hat{e}_z$ that satisfies incompressibility $\mathrm{div}(\bar{v}) = 0$. The b.c. will be $v_z(R) = 0$ (no slip).

Since $\frac{Dv_z}{Dt} = \underbrace{\frac{\partial v_z}{\partial t}}_{\text{steady state}} + (\bar{v}\cdot\mathrm{grad})\, v_z = \cancel{v_r \partial_r v_z} + \cancel{\frac{v_\theta}{r}\partial_\theta v_z} + \cancel{\partial_z v_z} = 0$

the z-component of the Navier-Stokes equation simplifies to

$$\nu \nabla^2 v_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \quad \Rightarrow \quad \nabla^2 v_z = \frac{\Delta p}{\eta \ell} \qquad \text{where } \nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\Delta p}{\eta\ell} \rightarrow \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\Delta p}{\eta\ell}r \rightsquigarrow r\frac{dv_z}{dr} = -\frac{\Delta p}{2\eta\ell}r^2 + C_1 \rightsquigarrow \frac{dv_z}{dr} = -\frac{\Delta p}{2\eta\ell}r + \frac{C_1}{r}$$

and finally $\underline{v_z(r) = -\frac{\Delta p}{4\eta\ell}r^2 + C_1 \log r + C_2}$

but $v_z(r\to 0)$ must remain finite $\Rightarrow C_1 = 0$; by the b.c. $v_z(R) = -\frac{\Delta p}{4\eta\ell}R^2 + C_2 = 0$

$$\Rightarrow C_2 = \frac{\Delta p}{4\eta\ell}R^2 \quad \Rightarrow v_z(r) = \frac{\Delta p}{4\eta\ell}\left(R^2 - r^2\right) \quad \left(= \frac{\Delta p R^2}{4\eta\ell}\left(1-\tilde{r}^2\right) \text{ with } \tilde{r} = r/R \text{ normalized radius}\right)$$

We find again a parabolic velocity profile, with a maximum on the axis and mean velocity (by which we can get the mass flow rate $\dot{m} = \rho\pi R^2 \tilde{v}_z$)

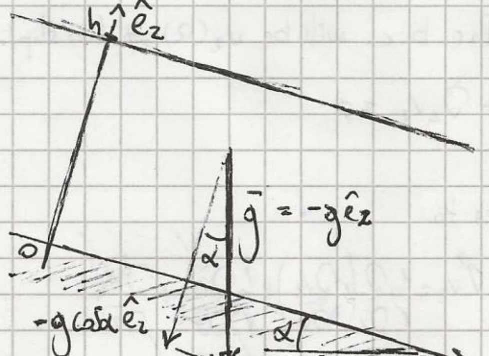
$$\tilde{v}_z = \frac{1}{\pi R^2}\int_0^R v_z(r)\, 2\pi r\, dr = \frac{\Delta p}{2\eta\ell R^2}\int_0^R \left(R^2 r - r^3\right)dr = \frac{\Delta p}{2\eta\ell R^2}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{\Delta p R^2}{8\eta\ell}$$

The shear stress on the wall is $-\sigma_{rz} = -\eta\left(\partial_r v_z + \cancel{\partial_z v_r}\right)\Big|_{r=R} = -\eta\,\partial_r v_z\Big|_{r=R} = \frac{\Delta p R}{2\eta\ell}$

(the wall normal vector is $-\hat{e}_r$)

Flow along an inclined plane

We consider here a viscous fluid layer of height h, flowing on top of a bottom with a certain slope – angle α with respect to the horizontal plane. Therefore here it is gravity that sets the fluid in motion. We are looking for steady-state solutions to the viscous and incompressible flow. To ease our calculations, we shall have a cs with $\hat{e}_x$ parallel to the bottom and $\hat{e}_z$ along the fluid layer thickness, with the usual y-invariance. Thus our



guess solution shall be $\underline{\bar{v} = v_x(z)\,\hat{e}_x}$

with b.c. ⊙ $v_x(z=\phi) = \phi$ (no slip on the bottom)

⊙ free surface at $z=h$ (air, negligible ρ on top) $\Rightarrow$ p on the surface should be p_0 atmospheric pressure, i.e. continuity of normal stress requires

$$\underline{\sigma_{zz} = -p = -p_0}$$

while no friction, i.e. no shear stress, should occur $\Rightarrow$

$$\underline{\sigma_{zx} = \eta\,\partial_z v_x\big|_{z=h} = \phi}$$

⊙ The component along $\hat{e}_z$ of the Navier-Stokes eq. reads

$$\underbrace{\frac{Dv_z}{Dt} = \phi}_{v_z=\phi} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 v_z - g\cos\alpha \Rightarrow \frac{dp}{dz} = -\rho g\cos\alpha \quad (p \text{ independent on } x)$$

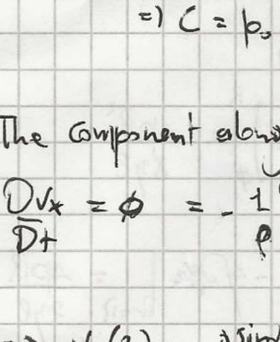
$$\Rightarrow p(z) = -\rho g\cos\alpha\, z + C; \quad \text{with the b.c. } p(h) = -\rho g\cos\alpha\, h + C = p_0$$

$$\Rightarrow C = p_0 + \rho g\cos\alpha\, h \Rightarrow \underline{p(z) = p_0 + \rho g\cos\alpha\,(h-z)}$$

⊙ The component along $\hat{e}_x$ of the Navier-Stokes eq. reads

$$\frac{Dv_x}{Dt} = \phi = -\frac{1}{\rho}\underbrace{\frac{\partial p}{\partial x}}_{\phi} + \nu\nabla^2 v_x + g\sin\alpha \rightsquigarrow \nu\frac{d^2 v_x}{dz^2} = -g\sin\alpha$$

$$\Rightarrow v_x(z) = -\frac{g\sin\alpha}{2\nu}z^2 + D_1 z + D_2 \quad ; \text{ the b.c. } v_x(\phi) = D_2 = \phi$$



and the b.c. $\partial_z v_x\big|_{z=h} = -\frac{g\sin\alpha}{\nu}h + D_1 = \phi \Rightarrow D_1 = \frac{g\sin\alpha}{\nu}h$

yield the solution

$$v_x(z) = \frac{g\sin\alpha}{2\nu}(2hz - z^2) \quad \left(= \frac{g\sin\alpha\, h}{2\nu}(2\tilde{z} - \tilde{z}^2) \text{ with } \tilde{z} \doteq z/h \right)$$

with friction on the bottom $\sigma_{zx} = \eta\,\partial_z v_x\big|_{z=\phi} = g\sin\alpha\, h/\nu$

Couette flow

While the Poiseuille flow is pressure-induced, the Couette flow is drag-induced, i.e. the fluid is set in motion by the movement of the domain's boundaries, dragging the fluid behind by friction.

Plane geometry

We imagine a steady-state incompressible viscous flow between two planes parallel to the horizontal plane, say $z=0$ and $z=h$, unbounded in x and y. While the bottom boundary is fixed, the top one is moving at constant velocity $\bar{v} = u\hat{e}_x$. Thus by x- and y-invariance we look for a solution $\bar{v} = v_x(z)\hat{e}_x$ (satisfying $\mathrm{div}(\bar{v}) = 0$) and we must require b.c. $v_x(0) = 0$, $v_x(h) = u$.

The x-component of the Navier–Stokes eq. reads

$$\frac{Dv_x}{Dt} = 0 = \nu \nabla^2 v_x - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{=0}$$

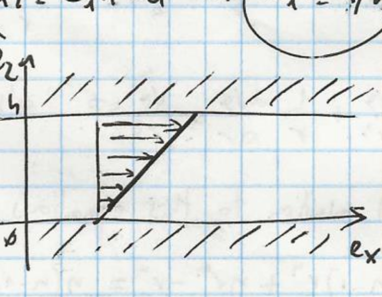
$=0$ no applied pressure gradient in x; only hydrostatic equilibrium along z

$$\Rightarrow \frac{d^2 v_x}{dz^2} = 0 \rightsquigarrow \frac{dv_x}{dz} = C_1 \rightsquigarrow v_x(z) = C_1 z + C_2$$

and imposing the b.c. $v_x(0) = C_2 = 0$; $v_x(h) = C_1 h = u \Rightarrow C_1 = u/h$

we get

$$v_x(z) = u\frac{z}{h} \quad \text{linear profile}$$



The stress tensor $\sigma_{ij} = \eta(\partial_j v_i + \partial_i v_j) - p\delta_{ij}$ reads

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p & 0 & \eta u/h \\ 0 & -p & 0 \\ \eta u/h & 0 & -p \end{pmatrix}$$

and in $z=0$ the shear stress on the wall (with normal unit vector $+\hat{e}_z$) is $\sigma_{zx} = \eta u/h$ (positive, the fluid pushes along $+\hat{e}_x$ and the fixed boundary opposes a resistance), while in $z=h$ the shear stress (being the wall's normal $-\hat{e}_z$) is $-\sigma_{zx} = -\eta u/h$ (negative, as the wall is moving along $+\hat{e}_x$ and trying to drag along the fluid, while the latter opposes a resistance against $\hat{e}_x$).

Notice there is a vorticity $\mathrm{curl}\,\bar{v} = \partial_z v_x\, \hat{e}_y = \frac{u}{h}\hat{e}_y$.

Cylindrical geometry

We consider the case of a flow between two coaxial cylinders of radii R_1, R_2 ($R_1 < R_2$) rotating with angular velocities Ω_1, Ω_2 respectively and infinite length (no finite-end effects); we look for a solution (considering ϑ-, z-invariance)

$\bar{v} = v_\vartheta(r)\,\hat{e}_\vartheta$ compliant with incompressibility $\operatorname{div}\bar{v} = \phi$.

We write the radial component of the Navier-Stokes eq.:

$$\cancel{\frac{\partial v_r}{\partial t}} + \underbrace{\left[(\bar{v}\cdot\text{grad})\bar{v}\right]_r}_{v_r \partial_r v_r = \phi} - \frac{v_\vartheta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\cancel{\nabla^2 v_r} - \cancel{\frac{2}{r}\frac{\partial v_\vartheta}{\partial \vartheta}} - \cancel{\frac{v_r}{r^2}}\right]$$

$\Rightarrow \dfrac{v_\vartheta^2}{r} = \dfrac{1}{\rho}\dfrac{\partial p}{\partial r}$ ← here we can see that there is v_ϑ^2/r centripetal acceleration, responsible for the curved trajectory of fluid elements ($p(r)$ can be obtained after finding the solution $v_\vartheta(r)$).

Now the ϑ-component of the Navier-Stokes eq. reads

$$\cancel{\frac{\partial v_\vartheta}{\partial t}} + \underbrace{(\bar{v}\cdot\text{grad})v_\vartheta}_{\frac{v_\vartheta}{r}\frac{\partial v_\vartheta}{\partial \vartheta}} + \cancel{\frac{v_r v_\vartheta}{r}} = -\cancel{\frac{1}{\rho r}\frac{\partial p}{\partial \vartheta}} + \nu\left[\nabla^2 v_\vartheta + \cancel{\frac{2}{r}\frac{\partial v_r}{\partial \vartheta}} - \frac{v_\vartheta}{r^2}\right]$$

$\longrightarrow$ ϑ-invariance

$$\Rightarrow \nabla^2 v_\vartheta - \frac{v_\vartheta}{r} = \phi \quad \rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\vartheta}{dr}\right) - \frac{v_\vartheta}{r^2} = \phi$$

$\rightsquigarrow \dfrac{d^2 v_\vartheta}{dr^2} + \dfrac{1}{r}\dfrac{dv_\vartheta}{dr} - \dfrac{v_\vartheta}{r^2} = \phi$ also written as $\boxed{r^2\dfrac{d^2 v_\vartheta}{dr^2} + r\dfrac{dv_\vartheta}{dr} - v_\vartheta = \phi}$ Cauchy-Euler eq.

General solution to this equations are in the form r^n; plugging it in,

$$n(n-1)r^n + nr^n - r^n = n(n-1)r^n + (n-1)r^n = (n+1)(n-1)r^n = (n^2-1)r^n = \phi$$

satisfied only with $n = \pm 1$ $\Rightarrow$

$$v_\vartheta(r) = ar + b/r$$

Let us apply now the b.c.: ① $v_\vartheta(R_1) = aR_1 + b/R_1 = \Omega_1 R_1$

② $v_\vartheta(R_2) = aR_2 + b/R_2 = \Omega_2 R_2$

With the operation $R_2\cdot② - R_1\cdot①$ we make b disappear and get

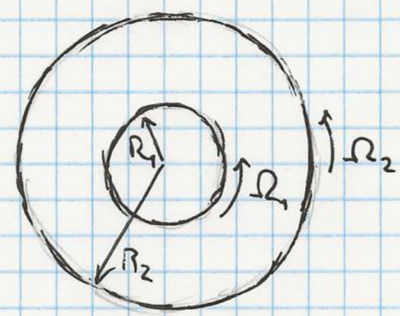
$$a(R_2^2 - R_1^2) = \Omega_2 R_2^2 - \Omega_1 R_1^2 \Rightarrow \boxed{a = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}}$$

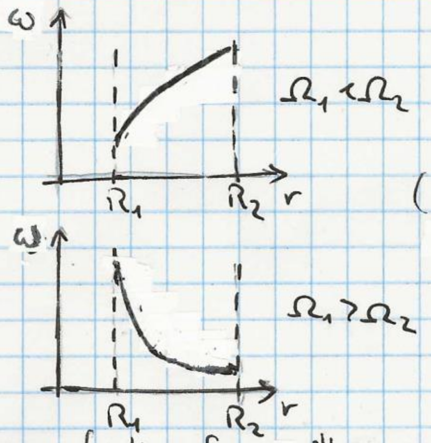
while with ①/R_1 − ②/R_1 we eliminate a and get $b(1/R_1^2 - 1/R_2^2) = \Omega_1 - \Omega_2$

$$\Rightarrow \boxed{b = \frac{(\Omega_1 - \Omega_2)R_1^2 R_2^2}{R_2^2 - R_1^2}}$$

$$\boxed{v_\vartheta(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\, r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}\, \frac{1}{r}}$$

(angular velocity shall be $\omega(r) = \frac{v_\vartheta(r)}{r} = a + b/r^2$)





(if $\Omega_1 = \Omega_2 \Rightarrow = \omega$, rigid rotation)

At the walls in R_1, R_2 the fluid exerts a friction force with a resulting moment (the two moments shall be equal and opposite). With the stress tensor, we can write

$$\sigma_{r\vartheta}\Big|_{r=R_2} \cdot R_2 = \text{moment per unit surface exerted by the } R_2 \text{ cylinder on the fluid } (\hat{e}_r \text{ normal})$$

$$= \eta\left(\cancel{\frac{1}{r}\frac{\partial v_r}{\partial \vartheta}} + \frac{\partial v_\vartheta}{\partial r} - \frac{v_\vartheta}{r}\right)\Big|_{r=R_2} \cdot R_2 = \eta\left(a - b/r^2 - a - b/r^2\right)\Big|_{r=R_2} R_2 =$$

$$= -2\eta b/R_2 = -\frac{2\eta(\Omega_1 - \Omega_2) R_1^2 R_2}{R_2^2 - R_1^2}$$

and multiplying by $2\pi R_2$ we have the moment per unit length of the cylinder

$$\tau_2 = \frac{-4\pi\eta(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

(positive if $\Omega_2 > \Omega_1$; see here that 1 and 2 can be swapped so $\tau_1 = -\tau_2$ as one should consider $-\sigma_{r\vartheta}\big|_{R_1}$)

Such a setup can be used to measure viscosity: By setting $\Omega_2 \neq 0$ ~~and~~ having the inner cylinder connected to a torsion spring, one can measure the torque τ_1 at the equilibrium condition $\Omega_1 = \emptyset$ and invert the expression of τ_1 for η (<u>Couette viscometer</u>).