Similarity laws and dimensionless numbers

When does a laboratory experiment reproduce another phenomenon, be it natural or artificial, occurring at different scales (spatial and time scales, for instance, as well as velocity, or force values)? In other words, how can we build a model correctly describing the physics we are interested in?

We can do such thing thanks to the concept of similarity and by manipulating the equations of motion so that they become dimensionless. By applying this technique we can obtain dimensionless parameters that contain the fundamental physical information of the system of interest, and we can identify a hierarchy in the terms that make up the equations of motion (so that we may possibly simplify them by cutting down negligible terms).

In the following we shall define various sorts of similarity and get to a dimensionless form of the Navier-Stokes equation.

⊙ Geometric similarity

Two flows are geometrically similar if their domains, and thus specifically the boundaries that identify them, transform one into each other by means of Euclidean (rigid) transformations, i.e.

- rototranslations (proper rigid transformations);
- reflections;
- isotropic expansions (shape-preserving similarity transformations)/contractions.

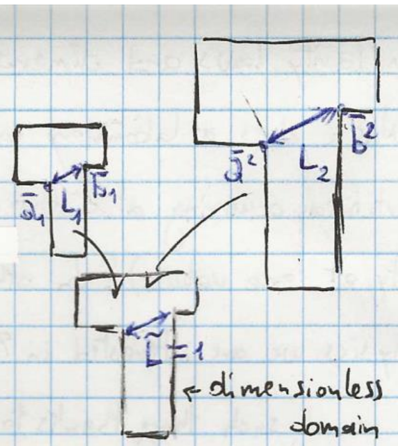
The first two types are not particularly interesting (unless there is a preferential direction, e.g. dictated by gravity - which breaks isotropy), so we focus on isotropic expansions and contractions, i.e. transformations that preserve angles $\Rightarrow$ geometric shapes.

Let us consider two points of coordinates $\bar{a}^1, \bar{b}^1$ in the flow ①, placed at a distance L_1; we shall choose them carefully, e.g. two boundary points at a characteristic distance (L_1 might be representative of an important length scale of the domain);

Then we shall consider $\bar{a}^2, \bar{b}^2$ in flow ②, which correspond to the pair $\bar{a}^1, \bar{b}^1$ in flow ① by geometric similarity (see figure);

We will get similarity transformations such that

$$\frac{1}{L_1}\bar{x}^1 = \tilde{x}' = \frac{1}{L_2}\bar{x}^2$$



that is to say any pair of corresponding point $\bar{x}^1, \bar{x}^2$ in the two domains is transformed into the same point $\tilde{x}'$ of a domain with dimensionless coordinates. In particular the two boundaries become the same boundary and the passage from one flow to the other for points $\bar{x}^2 \to \bar{x}^1$ takes place through the transformation $\bar{x}^1 = \frac{L_1}{L_2}\bar{x}^2$.

⊙ Kinematic similarity

Two flows are kinematically similar if:

a) They are geometrically similar, hence two geometrically corresponding points $\bar{x}^1, \bar{x}^2$ obey the transformation $\bar{x}^1/L_1 = \bar{x}^2/L_2 = \tilde{x}'$;

b) Velocities $\bar{v}^1(\bar{x}^1)$ and $\bar{v}^2(\bar{x}^2)$ observed in the two geometrically similar points obey another proportionality expression themselves, $\bar{v}^1(\bar{x}^1) = a\,\bar{v}^2(\bar{x}^2)$. This expression can be written considering two (carefully chosen) positions $\bar{a}^1, \bar{a}^2$ in the two flows ① and ② with velocities of magnitude U_1, U_2 respectively ($U_1 = |\bar{v}^1(\bar{a}^1)|$, $U_2 = |\bar{v}^2(\bar{a}^2)|$).

Then the velocity of $\bar{x}^2$ in flow ② transforms into the velocity of the geometrically similar point $\bar{x}^1$ in flow ① through the transformation

$$\bar{v}^1(\bar{x}^1) = \frac{U_1}{U_2}\,\bar{v}^2(\bar{x}^2) = \frac{U_1}{U_2}\,\bar{v}^2(\bar{x}_1 L_2/L_1)$$

The two flows can be reduced to the same dimensionless flow, with dimensionless velocity $\tilde{v}'$ and dimensionless coordinate $\tilde{x}'$

$$\tilde{v}'(\tilde{x}') = \frac{1}{U_1}\,\bar{v}^1(\underbrace{L_1\tilde{x}'}_{\bar{x}^1}) = \frac{1}{U_2}\,\bar{v}^2(\underbrace{L_2\tilde{x}'}_{\bar{x}_2})$$

⊙ Similar problems

Two problems are said to be similar if

a) They are geometrically similar;

b) They are kinematically similar on their boundaries (i.e. the condition $\frac{\bar{v}^1(\bar{x}^1)}{U_1} = \frac{\bar{v}^2(\bar{x}^2)}{U_2}$ is required only for boundary points).

If a), b) are satisfied, the two problems can be reduced to the same one (both in terms of domain and of b.c., which are actually expressing velocity at the boundary points) once they are rescaled to dimensionless coordinates and velocities.

Are these conditions enough to identify these "similar problems" as kinematically similar flows? We can claim it if they are proven to be <u>dynamically similar.</u>

⊙ Dynamic similarity

The concept of dynamic similarity is made clear and apparent if we process the set of quantities involved in a system's dynamics and make them dimensionless. Let us consider a system ① we would like to find a solution for, i.e. we would need to solve its Navier-Stokes equation. As a first example, let us say it is a steady-state problem without external volume forces. In order to obtain dimensionless (= normalized) variable we choose suitable values of length L_1 (a characteristic length, e.g. the diameter of a pipe if our problem is a flow in a pipe, or the distance between a pair of relevant points of the boundary) and velocity U_1 (a characteristic speed like the speed of a body within the flow, or the speed of an average mass flow rate). We can write dimensionless variables

$$\bar{x}' = \frac{1}{L_1}\bar{x} \quad ; \quad \bar{v}' = \frac{1}{U_1}\bar{v}(L_1\bar{x}') \quad ; \quad p' = \frac{1}{\rho_1 U_1^2}p^1(L_1\bar{x}')$$

for the Navier-Stokes eq. with its b.c. (and ρ_1, ν_1 known data for the problem):

$$\begin{cases} (\bar{v}(\bar{x})\cdot \text{grad})\,\bar{v}(\bar{x}) = -\frac{1}{\rho_1}\text{grad}\, p(\bar{x}) + \nu_1 \nabla^2 \bar{v}(\bar{x}) \\ \bar{v}(\bar{x})\Big|_{S_1} = \bar{v}_0^1(\bar{x}) \quad \leftarrow \text{b.c.: } \bar{v}_0^1(\bar{x}) \text{ velocity field assigned } \forall \bar{x} \in S_1 = \partial R_1 \text{, boundary of the domain } R_1 \end{cases}$$

We can say about differential operators that

$$\text{grad}' = \frac{1}{L_1}\text{grad}, \quad \nabla'^2 = \frac{1}{L_1^2}\nabla^2 \quad \text{since} \quad \underline{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x'_j}\frac{\partial x'_j}{\partial x_i} = \frac{\partial}{\partial x'_j}\frac{\partial}{\partial x_i}\left(\frac{x_j}{L_1}\right) = \frac{1}{L_1}\frac{\partial}{\partial x'_j}\delta_{ij} = \underline{\underline{\frac{1}{L_1}\frac{\partial}{\partial x'_i}}}$$

so if we plug in the dimensionless quantities,

$$\left(U_1\bar{v}'\cdot\frac{1}{L_1}\text{grad}'\right)\cdot U_1\bar{v}'(\bar{x}') = -\frac{1}{\rho_1}\frac{1}{L_1}\text{grad}'(\rho_1 U_1^2 p') + \nu_1\frac{1}{L_1^2}\nabla'^2(U_1\bar{v}'(\bar{x}'))$$

$$\Rightarrow \frac{U_1^2}{L_1}\bar{v}'\cdot\text{grad}'\,\bar{v}'(\bar{x}') = -\frac{U_1^2}{L_1}\text{grad}\,p'(\bar{x}') + \frac{\nu_1 U_1}{L_1^2}\nabla'^2(\bar{x}') \quad \text{and dividing by } U_1^2/L_1$$

$\bar{v}' \cdot \mathrm{grad}' \, \bar{v}'(\bar{x}') = -\mathrm{grad}' \, p'(\bar{x}') + \frac{\nu_1}{U_1 L_1} \nabla'^2 \bar{v}'(\bar{x}')$;

if we make dimensionless the b.c. too $\cancel{U_1} \bar{v}'(\bar{x}')\Big|_{S'} = \cancel{U_1} \bar{v}_0'(\bar{x}')$ with $S' = S/L_1$ dimensionless boundary, and by defining

$$\boxed{Re \doteq UL/\nu = \rho UL/\eta}$$ Reynolds number (a dimensionless parameter itself)

we finally get to stating our problem as

$$\begin{cases} \bar{v}'(\bar{x}') \cdot \mathrm{grad}' \, \bar{v}'(\bar{x}') = -\mathrm{grad}' \, p'(\bar{x}') + \frac{1}{Re_1} \nabla'^2 \bar{v}'(\bar{x}') \\ \bar{v}'(\bar{x}')\Big|_{S'} = \bar{v}_0'(\bar{x}') \quad \text{b.c.} \end{cases}$$ dimensionless Navier–Stokes eq.

The purpose of doing all this is finally apparent if we look at this form of the problem: If we had another problem ②, and we rescaled it using characteristic parameters L_2, U_2 starting from geometrically similar points with respect to ①, we would get

a) A dimensionless Navier-Stokes eq. with identical b.c. if

- there is a geometric similarity (① and ② have the same dimensionless b.c.),
- there is a kinematic similarity at the b.c. (① and ② have the same dimensionless velocity field at the boundary),

which are the conditions for similar problems;

b) The same Navier-Stokes eq., in addition to the same b.c., and therefore the same dimensionless solution for ① and ②, provided that $\boxed{Re_1 = Re_2}$ – notice that characteristic parameters and the physical content of the problems is contained in Re only.

We conclude that the problems are dynamically similar, and that the kinematic similarity we requested only on the boundary extends to the whole domain. Practically, all dynamically similar problems with the same Reynolds number, are solved by solving the same model problem; real quantities, with dimensions, can be obtained for each problem inverting the rescaling process, and they end up being functions of Re:

$\bar{v}(\bar{x}) = U \bar{v}'(\bar{x}/L, Re)$ from $\bar{v}' = \bar{v}'(\bar{x}', Re)$

$p(x) = \rho U^2 p'(\bar{x}/L, Re)$ from $p' = p'(\bar{x}', Re)$ and so on for all derived quantities (e.g., a drag force F can be $F = \rho U^2 A F'(Re)$ with A characteristic area).

Froude number (Navier-Stokes eq. in a gravitational field)

If on the right-hand side of the Navier-Stokes eq. we have a volume force field $\bar{g}$ like gravity, $\bar{g} = -g\hat{e}_z$, in the rescaling procedure we have to divide by U^2/L (indeed an acceleration, of force per unit mass in dimensional terms) and we get the term $-g\frac{L}{U^2}\hat{e}_z$; we define $\boxed{Fr \doteq U/\sqrt{gL} \quad \text{Froude number}}$

and the dimensionless N.-S. eq. is

$$(\bar{v}' \cdot \text{grad}')\,\bar{v}' = -\text{grad}\,p' + \frac{1}{Re}\nabla'^2\bar{v}' - \frac{1}{Fr^2}\hat{e}_z$$

Here dynamic similarity is guaranteed for problems featuring the same Re and F numbers.

Strouhal number (time-dependent N.-S. equation)

In time-dependent problems two cases are possible.

a) There is no independent characteristic time scale of the flow, i.e. a time that cannot be expressed as a function of other characteristic scales of the problem. Then we can build up a time scale starting from characteristic L and U, so that $\tau \doteq L/U$, expressing the spatial distance covered by the flow at typical velocity. With the usual rescaling operation,

$$\bar{x}' = \frac{1}{L}\bar{x}; \quad \bar{v}'(\bar{x}') = \frac{1}{U}\bar{v}(L\bar{x}'); \quad p'(\bar{x}') = \frac{1}{\rho U}p(\bar{x}'); \quad t' = \frac{1}{\tau}t = \frac{U}{L}t$$

and the added term on the left-hand side of the N.-S. eq. is

$$\frac{\partial \bar{v}}{\partial t} = \frac{U}{L}\frac{\partial}{\partial t'}(U\bar{v}') = \frac{U^2}{L}\frac{\partial \bar{v}'}{\partial t'}$$ so that dividing all terms by the usual U^2/L,

$$\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}' \cdot \text{grad}')\,\bar{v}' = -\text{grad}\,p' + \frac{1}{Re}\nabla'^2\bar{v}' + \frac{1}{Fr^2}\hat{e}_z$$

There is no significant change in the N.-S. eq.: since we added no further characteristic scale for the flow, the description of its dynamics is still enclosed in Re, Fr only.

b) An intrinsic time scale τ exists (e.g., the period of an oscillatory motion, a damping time, etc.), and time is rescaled through it:

$t' = \frac{1}{\tau}t \Rightarrow \frac{\partial \bar{v}}{\partial t} = \frac{U}{L}\frac{\partial \bar{v}'}{\partial t'} \Rightarrow$ dividing the N.-S. eq. by U^2/L we get

$$\frac{L}{U\tau}\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}'\cdot \mathrm{grad}')\,\bar{v}' = -\mathrm{grad}'p' + \frac{1}{Re}\nabla'^2\bar{v}' \left(+\frac{1}{Fr^2}\hat{e}_z\right)$$

and we define $\boxed{Sr \doteq \frac{L}{U\tau}}$ Strouhal number, hence

$$Sr\,\frac{\partial \bar{v}'}{\partial t} + (\bar{v}'\cdot \mathrm{grad}')\,\bar{v}' = -\mathrm{grad}'p' + \frac{1}{Re}\nabla'^2\bar{v}' \left(+\frac{1}{Fr^2}\hat{e}_z\right)$$

and dynamic similarity requests equal values of Sr, Re, Fr for all problems.

Use of dimensionless numbers to establish a hierarchy in the N.-S. equation

Normalization by the use of suitable $L, U(, \tau)$ allows us to directly compare the various terms in the N.-S. equation, first of all the advective term $(\bar{v}\cdot\mathrm{grad})\bar{v}$ and the viscous term $\nu\nabla^2\bar{v}$. Proper rescaling indeed means that these terms become of order 1, i.e. close to unity as we divide $\bar{x}$ with L, $\bar{v}$ with U and also rescale derivatives; the ratio of these two terms is hence completely expressed by Re:

$$\frac{\text{advective term}}{\text{viscous term}} = \frac{|(\bar{v}'\cdot\mathrm{grad}')\bar{v}'|}{\frac{1}{Re}|\nabla'^2\bar{v}'|} = Re$$

In other words (but equivalently) the fact that these terms are comparable (of the same order) can be written in terms of real quantities (with dimensions) as

$$(\bar{v}\cdot\mathrm{grad})\,\bar{v} \sim \nu\nabla^2\bar{v} \quad \text{and using } U, L$$

$$U\bar{v}'\frac{1}{L}U\bar{v}' = \frac{U^2}{L}(\bar{v}'^2) \sim \nu\frac{1}{L^2}U(\bar{v}') \;\leadsto\; Re = \frac{UL}{\nu} \sim 1$$

order 1

We conclude that the two terms are comparable (none can be neglected) if $Re \approx 1$.
We can explore the meaning of $Re \neq 1$, i.e. small and large Re limits.

* $Re \gg 1$: The viscous term is negligible with respect to the advective (inertial) one, and the N.-S. eq. reduces to Euler's equation, as viscosity does not come into play. Careful, though: This fact may be fine "in the open", in the bulk of the flow, far from boundaries and bodies, while in the vicinity of solid objects, any real fluid will experience a no-slip condition and high velocity gradients occur in the flow $\Rightarrow$ second-order derivatives in $\nu\nabla^2\bar{v}$ will make the viscous term absolutely not negligible. Once again, the ideal fluid approximation can be acceptable, but not in the boundary layer.
* It can happen that $Re \gg 1$ is not a global condition, but locally things can work out diffe-

rently. While a global Re may have been defined upon the choice of a certain L, we may be interested in observing the flow in a subdomain where a different typical length D exists, e.g., far from an obstacle of scale L, we are in an open region with scale D. If D > L we define a local Reynolds number

$$Re_D = \rho U D/\eta \quad > Re_{global} = \rho U L/\eta$$

such that Re_D could even be $\gg 1$ $\Rightarrow$ viscosity is locally negligible.

⊛ Re ≪ 1 (→ ∅): A problem dominated by viscosity, while the inertial term becomes negligible; eliminating this nonlinear term in $\bar{v}$ we get

$$\boxed{\rho \frac{\partial \bar{v}}{\partial t} = -\mathrm{grad}\, p + \eta \nabla^2 \bar{v}} \quad \text{called Stokes flow,}$$

whose steady-state form is $\eta \nabla^2 \bar{v} = \mathrm{grad}\, p$ and applying the curl,

$$\nabla^2 \mathrm{curl}\, \bar{v} = \phi \qquad (\nabla^2 \bar{\omega} = \phi)$$

This approximation is experimentally verified as fully acceptable, for instance in the case of a sphere moving in a fluid (or vice versa) when $Re < 0.1$, if Re is defined through the diameter of the sphere, and yield a fairly decent approximation up to $Re < 0.8$.

⊛ As with Re, we can interpret Fr by comparing advective and gravity terms:

$$(\bar{v}\cdot\mathrm{grad})\bar{v} \sim \bar{g}$$

$$\hookrightarrow \frac{U^2}{L}\underbrace{(\bar{v}')}_{\to \text{order-}1} \sim g \quad \to \text{comparable if } \frac{U^2}{gL} = Fr^2 \sim 1$$

In other words, $Fr = \left(\dfrac{\text{advective acceleration}}{\text{gravitational acceleration}}\right)^{1/2}$

⊛ Similar remarks hold for Sr, a comparison between advective term and intrinsic motion (oscillation, etc.):

$$Sr = \frac{\overset{U/\tau}{\text{intrinsic motion acceleration}}}{\underset{U^2/L}{\text{advective acceleration}}} \left(= \frac{|\partial \bar{v}/\partial t|}{|(\bar{v}\cdot\mathrm{grad})\bar{v}|} \right)$$