Damping of gravity waves

As much as we considered viscous damping in oscillatory motions induced by / onto solid objects in contact with the flow, we can try an evaluation of damping effects on oscillations at the fluid's free surface in the presence of viscosity, in the case of gravity waves.

Let us have gravity waves of amplitude a, wavelength λ, angular frequency ω at the free surface of a viscous fluid basin of large depth h ($h \gg \lambda, a$). The role of ℓ characteristic length scale of the obstacle is now taken by λ and we set the condition $\underline{\underline{a \ll \lambda}}$ - equivalent to the case ② discussed for the solid obstacle, so that we can claim there is a potential flow everywhere, except in a thin layer below the surface, where the velocity derivatives in the shear stresses must drop quite fast, in such a way that they vanish at the surface (where there is no friction), although the gradients are not as extreme as in contact with a solid surface, where $v_{\parallel}$ itself must vanish.

We know that mechanical energy dissipation reads

$$\frac{d}{dt}\bar{E}_{mech} = -\frac{1}{2}\eta\int_R\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)^2 d^3x$$

over the whole fluid domain R; as a matter of fact, we can restrict the integral to the potential flow region R_p, since

- The rotational flow region is small (thin superficial layer)
- The velocity gradients are not critically large in the rotational region

$\Big\}\Rightarrow$ negligible contribution

For a potential flow $\bar{v} = \operatorname{grad}\varphi \Rightarrow \frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial\varphi}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial\varphi}{\partial x_j} = \frac{\partial v_j}{\partial x_i}$

$$\Rightarrow \frac{d\bar{E}_{mech}}{dt} = -\frac{\eta}{2}\int_{R_p}\left(2\frac{\partial v_i}{\partial x_j}\right)^2 d^3x = -2\eta\int_{R_p}\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right)^2 d^3x$$

and since we know the solution to such flow $\varphi(x,z,t) = \varphi_0 e^{kz}\cos(kx-\omega t+\phi)$, let us calculate the time average of the integrand:

$$\langle(\partial_i\partial_j\varphi)^2\rangle = \langle(\partial_x\partial_x\varphi)^2 + (\partial_z\partial_z\varphi)^2 + (\partial_x\partial_z\varphi)^2 + (\partial_z\partial_x\varphi)^2\rangle =$$

$$= \langle k^4\varphi_0^2 e^{2kz}\cos^2(kx-\omega t+\phi) + k^4\varphi_0^2 e^{2kz}\cos^2(\dots) + k_0^4\varphi_0^2 e^{2kz}\sin^2(\dots) + k_0^4\varphi_0^2\sin^2(\dots)\rangle =$$

$$\left(\text{with } \langle\sin^2\rangle = \langle\cos^2\rangle = \frac{1}{2}\right) \quad = 4k^4\langle\varphi^2(x,z,t)\rangle$$

$$\Rightarrow \langle\dot{E}_{mech}\rangle = \left\langle\frac{d}{dt}E_{mech}\right\rangle = -8\eta k^4\int_{R_p}\langle\varphi^2\rangle d^3x$$

We know that in a small-oscillation regime, kinetic and potential energy have the same average value over a period (think, e.g., about the exchange of kinetic and potential energy in a pendulum); therefore we can calculate the average mechanical energy as

$$\langle E_{mech}\rangle = 2\langle E_K\rangle = \rho\int_{R_p}\langle v^2\rangle d^3x = \rho\int_{R_p}\langle \bar{v}\cdot\bar{v}\rangle d^3x = \rho\int_{R_p}\langle v_x^2+v_z^2\rangle d^3x =$$

$$= \rho\int_{R_p}\left[\left\langle\left(\frac{\partial\varphi}{\partial x}\right)^2\right\rangle + \left\langle\left(\frac{\partial\varphi}{\partial z}\right)^2\right\rangle\right]d^3x = \rho\int_{R_p}\left[\langle k^2\varphi_0^2 e^{2kz}\sin^2(kx-\omega t+\phi)\rangle + \langle k^2\varphi_0^2 e^{2kz}\cos^2(\dots)\rangle\right]d^3x =$$

$$= \rho\int_{R_p} k^2\, 2\langle\varphi^2\rangle d^3x = 2\rho k^2\int_{R_p}\langle\varphi^2\rangle d^3x \quad \Rightarrow \quad \langle E_{mech}\rangle = 2\rho k^2\int_{R_p}\langle\varphi^2\rangle d^3x$$

So if we define $2\gamma \doteq \langle \dot{E}_{mech}\rangle/\langle E_{mech}\rangle = 8\eta k^4/2\rho k^2 = 4\nu k^2 \quad (\gamma = 2\nu k^2)$

we have

$\langle E_{mech}\rangle = \langle E_{mech,0}\rangle e^{-2\gamma t}$ exponential decay of mechanical energy in time and

since $E_{mech} \propto a^2 \Rightarrow$ amplitude has a decay $a \sim e^{-\gamma t}$ with the inverse time constant γ; for a deep basin we also know the dispersion relation $\omega^2 = kg$

$$\Rightarrow \boxed{\gamma = 2\nu k^2 = 2\nu\,\omega^4/g^2}$$

For a shallow basin the situation is completely different and damping results to be dominated by friction against the bottom and the calculation, involving the theory of the boundary layer, yields $\boxed{\gamma = \frac{(\omega\nu/2)^{\frac{1}{2}}}{2h}}$

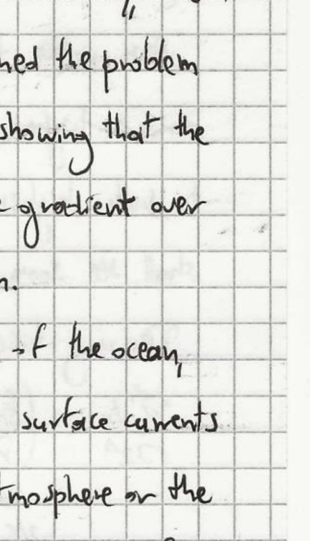
A dramatic case of a wave with $\lambda \gg h$ (i.e. where a shallow-basin description holds) is the tsunami: Its wavelength is $\lambda \sim 10-100$ km while the ocean depth $h \sim 4-5$ km only; so the propagation velocity is $v_g = v_f = \sqrt{gh} \sim 200$ m/s, quite large, and this is combined with the fact that the amplitude can grow from offshore values of $\sim 10-100$ cm to several meters on the shore*, yielding a devastating energy flux with destructive effects; furthermore, the damping time scale $1/\gamma$ is very large, and therefore a tsunami can go around the globe (more than once) before decaying off.

* = Even if increase of amplitude is accompanied by a decrease in speed by an order of magnitude at least ($v = \sqrt{gh}$), it still means tens of km/h!

Ekman layer

During his exploration of the Arctic region on the Fram ship at the end of the XIX century, Fridtjof Nansen (explorer, scientist and later Nobel Peace Prize laureate for his work as a diplomat and humanitarian) observed a weird phenomenon: Icebergs would not drift along the direction of the wind, but at an angle to the right of said direction. Back in Norway, Nansen told his colleague and physicist Vilhelm Bjerknes about this and Bjerknes assigned the problem to his student Vagn Walfrid Ekman, who solved the issue in his doctoral thesis, showing that the balance of the Coriolis force (i.e. the Earth's rotation), viscous drag and pressure gradient over a superficial layer of the sea is responsible for the behaviour observed by Nansen.

Two distinct occurrences of this phenomenon are observed: ① At the free surface of the ocean, due to the wind that not only can stimulate surface waves but also gives rise to surface currents when it maintains a steady intensity and direction); ② At the bottom of the atmosphere or the ocean, when the fluid motion exerts a viscous friction on a rough surface. We shall discuss the first type of problem.



We have the Earth rotating with angular frequency $\bar{\omega} = \omega \hat{e}_n$; calling $\varphi \in [-\pi/2; \pi/2]$ the latitude angle, there is a Coriolis force per unit mass $\bar{f}_c$ on a fluid element with velocity $\bar{v}$ that is

$\bar{f}_c = 2\bar{v} \times \bar{\omega}$, at a position of latitude φ, and defining

$f \doteq 2\omega \sin\varphi$ Coriolis parameter (or frequency),

if we consider a local cs with x, y on the horizontal plane and z local vertical direction, $\bar{f}_c$ is decomposed in its x, y components

$$\begin{cases} f_{cx} = f v_y \\ f_{cy} = -f v_x \end{cases}$$

Now let us linearize the Navier-Stokes equation, including the Coriolis force; linearization means neglecting the nonlinear advective term $(\bar{v} \cdot \mathrm{grad})\bar{v}$:

$$\begin{cases} \dfrac{\partial v_x}{\partial t} = f v_y - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \tilde{\nu}\dfrac{\partial^2 v_x}{\partial z^2} \\ \dfrac{\partial v_y}{\partial t} = -f v_x - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \tilde{\nu}\dfrac{\partial^2 v_y}{\partial z^2} \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \end{cases}$$

(since we are considering an x, y-invariant phenomenon like a uniform horizontal current, $\bar{v} = \bar{v}(z)$ with $v_z = 0$ $\Rightarrow (\bar{v} \cdot \mathrm{grad})\bar{v} = 0$, and $\nabla^2 \bar{v} = \partial^2 \bar{v}/\partial z^2$)

where, considering a steady-state phenomenon and the absence of horizontal pressure gradient, we simplify the system to

$$\begin{cases} \tilde{\nu} \dfrac{\partial^2 v_x}{\partial z^2} + f v_y = \emptyset \\ \tilde{\nu} \dfrac{\partial^2 v_y}{\partial z^2} - f v_x = \emptyset \end{cases}$$ (the z-component is simply the hydrostatic equilibrium)

Also notice that we write $\tilde{\nu}$ (or $\tilde{\eta} = \rho\tilde{\nu}$) instead of ν, and by that we mean that a much higher viscosity must be considered. Indeed the turbulent mixing of the flow gives rise to such higher "eddy viscosity" — and in turn, to a much thicker Ekman layer, as we shall see soon.

Deriving twice the first equation with respect to z and thus eliminating v_y we obtain $\dfrac{\partial^4 v_x}{\partial z^4} = -\left(\dfrac{f}{\tilde{\nu}}\right)^2 v_x$ and define $\varepsilon \doteq \sqrt{2\tilde{\nu}/|f|} = \sqrt{\tilde{\nu}/\omega \sin(|\varphi|)}$; the general solution is

$$v_x \sim e^{\pm z/\varepsilon} e^{\pm i z/\varepsilon}$$

where $e^{-z/\varepsilon}$ has to be excluded since v_x must remain finite at large depth ($z \to -\infty$). Hence the solution is a combination of oscillating functions with a superimposed damping for increasing depth. If we say the velocity at the surface ($z=\emptyset$) is $\bar{u}_0$ and we set up a cs where $\bar{u}_0 \parallel \hat{e}_x$, the solution can be written as

$$v_x = u_0 e^{z/\varepsilon} \cos(z/\varepsilon)$$ (we do not care about an absolute phase here)

and plugging it into the differential eq. $\tilde{\nu}\partial^2_{zz} v_x + f v_y = \emptyset$ we get, with some algebra,

$$f v_y = \frac{2\tilde{\nu} u_0}{\varepsilon^2} e^{z/\varepsilon} \sin(z/\varepsilon) \Rightarrow v_y = \frac{2\tilde{\nu}}{f} \frac{|f|}{2\tilde{\nu}} u_0 e^{z/\varepsilon} \sin(z/\varepsilon) = \frac{\sin|\varphi|}{\sin\varphi} u_0 e^{z/\varepsilon} \sin(z/\varepsilon).$$ Summarizing,

$$\begin{cases} v_x = u_0 e^{z/\varepsilon} \cos(z/\varepsilon) \\ v_y = u_0 \dfrac{\sin|\varphi|}{\sin\varphi} e^{z/\varepsilon} \sin(z/\varepsilon) \end{cases}$$

where we see that the velocity not only experiences a damping over a depth scale ε, but also, as a vector, has an angle ϑ with respect to $\bar{u}_0$ such that ($\vartheta = \emptyset$ for vector $\parallel \bar{u}_0 \Rightarrow \parallel \hat{e}_x$)

$$\frac{v_y}{v_x} = \tan\vartheta = \frac{\sin|\varphi|}{\sin\varphi} \tan(z/\varepsilon)$$

$\Rightarrow \vartheta = \pm z/\varepsilon$ angle depending on the depth and orientation ($\pm$) depending on the hemisphere (+/− → North / South). The velocity vector rotates and decreases in amplitude when going from the surface towards the Ocean's depth, substantially vanishing over a folding spatial scale ε ("Ekman spiral"). We can define an average velocity vector of the surface current in the layer of thickness ε as

$$\langle \bar{v} \rangle = \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} \bar{v}(z)\,dz \simeq \frac{1}{\varepsilon}\int_{-\infty}^{0} \bar{v}(z)\,dz \quad \left(\text{since anyway } \bar{v} \to 0 \text{ for } z < -\varepsilon\right); \Rightarrow$$

$$\langle v_x \rangle = \frac{1}{\varepsilon}\int_{-\infty}^{0} u_0 e^{z/\varepsilon}\cos(z/\varepsilon)\,dz = u_0\left[\frac{1}{2}e^{z/\varepsilon}\left[\sin(z/\varepsilon)+\cos(z/\varepsilon)\right]\right]_{-\infty}^{0} = u_0\,\frac{1}{2}\left[\underbrace{\sin(0)}_{0}+\underbrace{\cos(0)}_{1}\right] = \frac{1}{2}u_0$$

$$\langle v_y \rangle = \frac{1}{\varepsilon}\int_{-\infty}^{0} \frac{\sin|\varphi|}{\sin\varphi}u_0 e^{z/\varepsilon}\sin(z/\varepsilon)\,dz = \frac{\sin|\varphi|}{\sin\varphi}u_0\left[\frac{1}{2}e^{z/\varepsilon}\left[\sin(z/\varepsilon)-\cos(z/\varepsilon)\right]\right]_{-\infty}^{0} = -\frac{1}{2}\frac{\sin|\varphi|}{\sin\varphi}u_0$$

therefore $\langle \bar{v} \rangle$ has an angle θ' with respect to $\hat{e}_x$ /

$$\frac{\langle v_y \rangle}{\langle v_x \rangle} = \tan\theta' = -\frac{\sin|\varphi|}{\sin\varphi} = \mp 1 \quad \Rightarrow \quad \theta' = \mp \pi/4 \text{ depending on the hemisphere}$$

We can now also determine the relative orientation with respect to the wind; indeed the wind direction sets the direction of the shear stress at the surface; in components,

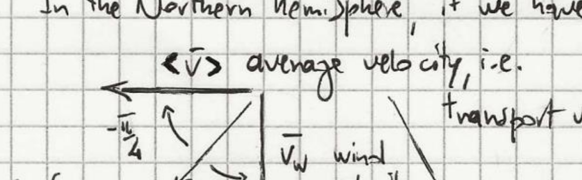
$$\sigma_{zx} = \tilde{\eta}\left(\frac{\partial v_x}{\partial z}\right)\bigg|_{z=0} = \tilde{\eta}u_0\left[\frac{1}{\varepsilon}e^{z/\varepsilon}\cos(z/\varepsilon) - \frac{1}{\varepsilon}e^{z/\varepsilon}\sin(z/\varepsilon)\right]\bigg|_{z=0} = \tilde{\eta}u_0/\varepsilon$$

$$\sigma_{zy} = \tilde{\eta}\left(\frac{\partial v_y}{\partial z}\right)\bigg|_{z=0} = \tilde{\eta}u_0\frac{\sin|\varphi|}{\sin\varphi}\left[\frac{1}{\varepsilon}e^{z/\varepsilon}\sin(z/\varepsilon) + \frac{1}{\varepsilon}e^{z/\varepsilon}\cos(z/\varepsilon)\right]\bigg|_{z=0} = \tilde{\eta}\frac{u_0}{\varepsilon}\frac{\sin|\varphi|}{\sin\varphi}$$

$$\Rightarrow \frac{\sigma_{zy}}{\sigma_{zx}} = \tan\theta_w = \frac{\sin|\varphi|}{\sin\varphi} = \pm 1 \quad \Rightarrow \quad \theta_w = \pm\pi/4 \text{ depending on the hemisphere}$$

where θ_w is the angle of the wind orientation with respect to $\hat{e}_x \parallel \bar{u}_0$.

In the Northern hemisphere, if we have a wind from the North we get



This is, for instance, the situation on the Californian coast, where the wind parallel to the coast (from North to South) causes an average current to the West, i.e. offshore, of the surface layer, with an ensuing upwelling: A rise to the surface of the deeper, cooler waters, with significant effects such as fog (by cooling of the atmosphere and humidity condensation) and increase of flora and fauna (as deep water is typically richer in nutrients than the depleted surface).