Material derivative of volume integrals

Given a continuum and a certain region of space $R(t)$ that is occupied by a part of such continuum at the time instant t, let us consider an extensive quantity whose value over $R(t)$ is $\mathcal{F}(t)$, and let us consider the quantity per unit volume associated to $\mathcal{F}(t)$, called $F(\bar{x}(t),t)$; in brief,

$$\mathcal{F}(t) = \int_{R(t)} F(\bar{x}(t),t)\, d^3x$$

Note that $\mathcal{F}(t)$ depends on t both explicitly and implicitly, since as we follow the motion of the continuum, $R(t)$ evolves in time as well. Then the time derivative of $\mathcal{F}(t)$ is a material derivative and special attention is advised. One can prove that

$$\frac{D}{Dt}\mathcal{F}(t) = \int_{R(t)} \Big[\frac{D}{Dt}F(\bar{x}(t),t) + \underbrace{F(\bar{x}(t),t)\,\mathrm{div}\,\bar{v}(\bar{x}(t),t)}_{\text{a term associated to the variation of } R \text{ through time}}\Big]\, d^3x = \int_{R(t)} \Big[\frac{\partial F}{\partial t} + \mathrm{div}\big(F(\bar{x},t)\,\bar{v}(\bar{x},t)\big)\Big] d^3x$$

Proof: By definition of time derivative and $\mathcal{F}(t)$

$$\frac{D}{Dt}\mathcal{F}(t) = \frac{1}{\Delta t}\lim_{\Delta t\to 0}\left[\int_{R(t+\Delta t)} F(\bar{x}(t+\Delta t), t+\Delta t)\, d^3x - \int_{R(t)} F(\bar{x}(t),t)\, d^3x\right]$$

If we call $\bar{x}' = \bar{x}(t+\Delta t)$, we can write, to first order in t, $\bar{x}' = \bar{x}'(\bar{x}) = \bar{x} + \bar{v}\Delta t$ and we can manipulate the first integral in the limit, performed over $R' = R(t+\Delta t)$ at $t+\Delta t$:

$$\int_{R(t+\Delta t)} F(\bar{x}(t+\Delta t), t+\Delta t)\, d^3x = \int_{R'} F(\bar{x}', t+\Delta t)\, d^3x' =$$ by changing the integration variable $\bar{x}' \to \bar{x}$

$$= \int_R F(\bar{x}'(\bar{x}), t+\Delta t)\,|J(x'|x)|\, d^3x = \int_R F(\bar{x} + \bar{v}(\bar{x},t)\Delta t, t+\Delta t)\,|J(x'|x)|\, d^3x$$

$\bar{x}'$ to first order as seen above

where $J(x'|x)$ is the Jacobian matrix of the transformation; again to first order

$$|J(x'|x)| = \det\big(J(x'|x)\big) = \det\left(\frac{\partial x_i}{\partial x_j}\right) = \det\left(\delta_{ij} + \frac{\partial v_i(\bar{x},t)}{\partial x_j}\Delta t\right) = 1 + \Delta t\, \mathrm{Tr}\left(\frac{\partial v_i(\bar{x},t)}{\partial x_j}\right) =$$

using $\det(\mathbb{1} + \phi \underline{\underline{A}}) = 1 + \phi\,\mathrm{Tr}(\underline{\underline{A}})$ since $\Delta t = \phi$ is infinitesimal

$$= 1 + \Delta t \frac{\partial v_i(\bar{x},t)}{\partial x_i} = 1 + \Delta t\, \mathrm{div}\big(\bar{v}(\bar{x},t)\big)$$

From the expression of the differential $DF(\bar{x}(t),t)$ we can write, always to first order,

$$F(\bar{x}(t+\Delta t), t+\Delta t) = F(\bar{x}(t),t) + \frac{D}{Dt}F(\bar{x}(t),t)\,\Delta t$$

Let us use all of this in the expression of $\tilde{F}(t)$'s total derivative:

$$\frac{D\tilde{F}(t)}{Dt} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{R(t+\Delta t)} F(\bar{x}(t+\Delta t), t+\Delta t)\,d^3x - \int_{R(t)} F(\bar{x}(t),t)\,d^3x\right] =$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{R(t)}\left[F(\bar{x}(t),t) + \Delta t\,\frac{DF}{Dt}(\bar{x}(t),t)\right]\left[1+\Delta t\,\mathrm{div}\left(\bar{v}(\bar{x}(t),t)\right)\right]d^3x - \int_{R(t)} F(\bar{x}(t),t)\,d^3x\right\} =$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{R(t)}\left[F(\bar{x}(t),t)\,\mathrm{div}(\bar{v}(\bar{x},t))\,\Delta t + \frac{DF}{Dt}(\bar{x}(t),t)\,\Delta t + \frac{DF}{Dt}(\bar{x}(t),t)\,\mathrm{div}(\bar{v}(\bar{x},t))(\Delta t)^2\right]d^3x\right\} =$$

order $(\Delta t)^2 \Rightarrow$ negligible

$$= \int_{R(t)}\left[\frac{DF}{Dt}(\bar{x}(t),t) + F(\bar{x}(t),t)\,\mathrm{div}\left(\bar{v}(\bar{x}(t),t)\right)\right]d^3x =$$ with the whole explicit expression of D/Dt

$$= \int_{R(t)}\left[\frac{\partial F}{\partial t}(\bar{x},t) + \underbrace{(\bar{v}\cdot\mathrm{grad})F(\bar{x},t) + F(\bar{x},t)\,\mathrm{div}\,\bar{v}(\bar{x},t)}_{\mathrm{div}(F\bar{v})}\right]d^3x \Rightarrow$$

$$\frac{D\tilde{F}(t)}{Dt} = \int_{R(t)}\left[\frac{\partial F}{\partial t}(\bar{x}(t),t) + \mathrm{div}\left(F(\bar{x}(t),t)\,\bar{v}(\bar{x}(t),t)\right)\right]d^3x$$

Reynolds transport theorem

⊙ This expression holds for any extensive quantity $\tilde{F}$ and its associated quantity per unit volume F. The first and fundamental result of physical relevance is using it for $F=\rho$ (mass density) which results in the continuity equation (i.e., mass conservation).

⊙ Using the divergence theorem, we can express the Reynolds transport theorem as

$$\frac{D\tilde{F}(t)}{Dt} = \frac{D}{Dt}\int_{R(t)} F(\bar{x}(t),t)\,d^3x = \int_{R(t)}\frac{\partial F}{\partial t}(\bar{x}(t),t)\,d^3x + \int_{\partial R(t)} F(\bar{x}(t),t)\,\bar{v}(\bar{x}(t),t)\cdot d\bar{S}$$

that is, the rate of change (time variation) of an extensive property $\tilde{F}$ is given by the sum of the rate of change of the associated property per unit volume, calculated over the control volume, and the flux of the quantity per unit volume across the control volume boundary.

Continuity equation – Incompressibility condition

If we take a certain portion of continuum enclosed in a region $R(t)$ at the time instant t, and then we follow it in its motion, we trivially know that its mass M will be conserved. Writing the mass M in terms of its quantity per unit volume i.e. the density $\rho(\bar{x},t)$ we can state

$$M(t) = \int_{R(t)} \rho(\bar{x}(t),t)\, d^3x = \text{constant},$$

i.e. since we follow the continuum element in $R(t)$ along its motion,

$$\frac{DM(t)}{Dt} = \phi \quad , \quad \text{i.e.} \quad \frac{D}{Dt}\int_{R(t)} \rho(\bar{x}(t),t)\, d^3x = \int_{R(t)} \left[\frac{\partial \rho}{\partial t}(\bar{x}(t),t) + \operatorname{div}\left(\rho(\bar{x},t)\,\bar{v}(\bar{x},t)\right)\right] d^3x = \phi$$

But since $R(t)$ is arbitrary, null integral implies null integrand, that is

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\bar{v}) = \phi} \qquad \text{CONTINUITY EQUATION}$$

, also written as

$$\frac{D\rho}{Dt} + \rho \operatorname{div}(\bar{v}) = \phi \qquad \text{or} \qquad \frac{1}{\rho}\frac{D\rho}{Dt} = -\operatorname{div}(\bar{v}) \qquad \text{or} \qquad \frac{1}{\upsilon}\frac{D\upsilon}{Dt} = \operatorname{div}(\bar{v})$$

where $\upsilon \doteq \frac{1}{\rho}$ is the specific volume (volume per unit mass)

From this we draw the (mechanical) incompressibility condition ($\rho = \text{constant} \overset{?}{\Leftrightarrow} D\rho/Dt = \phi$)

$$\boxed{\operatorname{div}(\bar{v}) = \phi}$$

Note: Let us use the rule for the derivative of integrals with

$$F(\bar{x},t) = 1 \quad \Rightarrow \quad \mathcal{F}(t) = \int_{R(t)} 1 \cdot d^3x = V \quad \text{volume of the region } R; \quad \Rightarrow$$

$$\frac{DV}{Dt} = \int_{R(t)} \Big[\underbrace{\cancel{\frac{\partial}{\partial t}(1)}}_{\phi} + \operatorname{div}(1\cdot\bar{v})\Big] d^3x = \int_{R(t)} \operatorname{div}(\bar{v})\, d^3x = \int_{\partial R(t)} \bar{v}\cdot\hat{n}\, da$$

with the divergence theorem

where $\partial R(t)$ is R's boundary, i.e. the closed surface that contains the region R, with local normal (outward-pointing) unit vector $\hat{n}$. This result tells us that the volume variation of a continuum element in time is given by the velocity flux through the boundary of the very element.

Integral and local form of the laws of physics

We often see the laws of physics written in two alternative forms, i.e. either locally or 'under the umbrella' of an integral. The link between this two forms is crucial and it is worth spending some effort to explicitly elucidate it. Of course, in the end, they must be equivalent.

As far as we are concerned, integral laws we will see share a structure: The equality between the total (material) derivative of an integral and, on the other side, another integral quantity.

Example: Let us write the first law of motion for a continuum element of density $\rho(\bar{x},t)$ enclosed within the region $R(t)$ and subjected to an overall resultant force $\bar{F}(t)$:

$$\frac{D}{Dt}\int_{R(t)} \rho(\bar{x},t)\bar{v}(\bar{x},t)\, d^3x = \bar{F}(t) \qquad (*)$$

$\bar{F}(t)$ has an associated force per unit mass $\bar{f}$, by integration of $\rho\bar{f}$ (that is a force per unit volume) over $R(t)$, $\bar{F}(t) = \int_{R(t)} \rho(\bar{x},t)\bar{f}(\bar{x},t)\, d^3x \Rightarrow$

$$\frac{D}{Dt}\int_{R(t)} \rho(\bar{x},t)\bar{v}(\bar{x},t)\, d^3x = \int_{R(t)} \rho(\bar{x},t)\,\bar{f}(\bar{x},t)\, d^3x$$

We will show a little below that since ρ obeys the continuity eq.,

$\frac{D}{Dt}\int_{R(t)} \rho\bar{v}\, d^3x = \int_{R(t)} \rho\frac{D\bar{v}}{Dt}\, d^3x$, $\Rightarrow$ the equality above becomes

$\int_{R(t)} \rho\frac{D\bar{v}}{Dt}\, d^3x = \int_{R(t)} \rho\bar{f}\, d^3x$ but since $R(t)$ is arbitrary, the integrand functions must be equal, $\Rightarrow$ (for $\rho\neq 0$)

$$\frac{D\bar{v}}{Dt} = \bar{f} \qquad (**)$$

and $(**)$ expresses locally the integral form $(*)$ of the first law of motion, using quantities per unit mass: $\bar{f}$ is the force per unit mass,

$\bar{v}$ is the linear momentum per unit mass.

This kind of relation, exemplified here for $\bar{v}$, has a general validity for any extensive quantity per unit mass g. The ρg is the quantity per unit volume and its volume integral yields the associated extensive quantity over a certain region of continuum; writing it derivative

$$\frac{D}{Dt}\int_R \rho g\, d^3x = \int_R \left[\frac{\partial}{\partial t}(\rho g) + \mathrm{div}(\rho g \bar{v})\right] d^3x =$$

$$= \int_R \left[\rho \frac{\partial g}{\partial t} + g\frac{\partial \rho}{\partial t} + g\,\mathrm{div}(\rho \bar{v}) + \rho \bar{v}\cdot \mathrm{grad}(g)\right] d^3x =$$

$$= \int_R \left\{ \underbrace{g\left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \bar{v})\right]}_{=\phi \text{ as imposed by continuity eq.}} + \rho\left[\frac{\partial g}{\partial t} + (\bar{v}\cdot\mathrm{grad})\, g\right]\right\} d^3x = \int_R \rho \frac{Dg}{Dt}\, d^3x$$

$\Rightarrow \displaystyle\frac{D}{Dt}\int_R \rho g\, d^3x = \int_R \rho \frac{Dg}{Dt}\, d^3x$ and if h quantity per unit mass exists such that

$$(***) \quad \frac{D}{Dt}\int_R \rho g\, d^3x = \int_R \rho h\, d^3x \quad \Rightarrow \quad \int_R \rho\frac{Dg}{Dt}\, d^3x = \int_R \rho h\, d^3x$$

$\Rightarrow$ the integrand functions must be equal (R is arbitrary) $\Rightarrow$ the integral equation $(***)$ is equivalent to the local law

$$\frac{Dg}{Dt} = h$$

Note: Along this calculation we obtain an interesting intermediate result, i.e. a relationship between ρg (quantity per unit volume) with its partial time derivative and g (the corresponding quantity per unit mass) and it total (material) derivative,

$$\frac{\partial}{\partial t}(\rho g) + \mathrm{div}(\rho g \bar{v}) = \underset{\substack{\uparrow \\ \text{using the continuity eq.}}}{\ldots} = \rho \frac{Dg}{Dt}$$

Material derivative of line integrals

Given a curve $\gamma(t)$ that is made out of points (point particles) of a continuum in motion, if we consider a quantity $f(\bar{x},t)$ then

$$\frac{D}{Dt}\int_{\gamma(t)} f(\bar{x},t)\,ds_i = \int_{\gamma(t)} \left(f\frac{\partial v_i}{\partial x_j} + \frac{D}{Dt} f\delta_{ij} \right) ds_j$$

where ds_i is the i-th component of the infinitesimal path $d\bar{x}$ along the curve, and v_i the i-th component of $\bar{v}$ velocity.

Proof: The curve γ can be described as a function taking values $\in \mathbb{R}^3$ parameterized by using a parameter α running through the interval of real values with limit points a and b - $\alpha \in [a,b]$. There may be a number of different parametrizations for γ.

So we choose a certain parametrization expressed as $\bar{x} = \bar{x}(\alpha)$, with $\alpha \in [a,b]$, that is a smooth function with values in $\mathbb{R}^3$, or components $x_i(\alpha) \in \mathbb{R}$. The line integral of the parameterized curve is hence a Riemann integral over $[a,b]$

$$\int_\gamma f(\bar{x})\,ds_i = \int_a^b f(\bar{x}(\alpha))\,\frac{dx_i(\alpha)}{d\alpha}\,d\alpha$$ with material derivative

$$\frac{D}{Dt}\int_{\gamma(t)} f(\bar{x},t)\,ds_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{\gamma(t+\Delta t)} f(\bar{x}(t+\Delta t), t+\Delta t)\,ds_i - \int_{\gamma(t)} f(\bar{x}(t), t)\,ds_i \right]$$

A detail sometimes left implicit, yet a fundamental one: We are choosing a suitable paramotrization, i.e. such that at the time instant t, $\bar{x} = \bar{x}(\alpha(t))$, and at time instant $t+\Delta t$ <u>the same value for α</u> will yield $\bar{x}(\alpha, t+\Delta t)$, that is exactly the evolution of the point (point particle) $\bar{x}(t)$ after a time interval Δt. If this is true, we can write to first order in Δt

$$\bar{x}(\alpha, t+\Delta t) = \bar{x}(\alpha,t) + \bar{v}(\bar{x}(\alpha,t))\Delta t$$

and manipulate the difference

$$\int_{\gamma(t+\Delta t)} f(\bar{x}, t+\Delta t)\,ds_i - \int_{\gamma(t)} f(\bar{x},t)\,ds_i =$$ by operating the parametrization

$$= \int_a^b f(\bar{x}(\alpha, t+\Delta t), t+\Delta t)\, \frac{dx_i(\alpha, t+\Delta t)}{d\alpha}\, d\alpha - \int_a^b f(\bar{x}(\alpha, t), t)\, \frac{dx_i(\alpha, t)}{d\alpha}\, d\alpha =$$

$$= \int_a^b f(\bar{x}(\alpha, t+\Delta t), t+\Delta t)\, \frac{\partial x_i(\alpha, t+\Delta t)}{\partial x_j(\alpha, t)} \cdot \frac{dx_j(\alpha, t)}{d\alpha}\, d\alpha - \int_a^b f(\bar{x}(\alpha, t), t)\, \frac{dx_j(\alpha, t)}{d\alpha}\, \delta_{ij}\, d\alpha =$$

$$= \int_a^b \left[f(\bar{x}(\alpha, t+\Delta t), t+\Delta t)\, \frac{\partial x_i(\alpha, t+\Delta t)}{\partial x_j(\alpha, t)} - f(\bar{x}(\alpha, t), t)\, \delta_{ij} \right] \frac{dx_j(\alpha, t)}{d\alpha}\, d\alpha =$$ (now we write everything at time t)

⊙ Now since by definition $\frac{D}{Dt} f(\bar{x}(\alpha, t), t) = \frac{1}{\Delta t} \left[f(\bar{x}(\alpha, t+\Delta t), t+\Delta t) - f(\bar{x}(\alpha, t), t) \right]$ (with Δt infinitesimal)

we can invert and write $f(\bar{x}(\alpha, t+\Delta t), t+\Delta t) = f(\bar{x}(\alpha, t), t) + \Delta t\, \frac{D}{Dt} f(\bar{x}(\alpha, t), t)$

⊙ $\frac{\partial}{\partial x_j(\alpha, t)}\, x_i(\alpha, t+\Delta t) = \frac{\partial}{\partial x_j(\alpha, t)} \left[x_i(\alpha, t) + v_i(\bar{x}(\alpha, t), t)\, \Delta t \right] = \delta_{ij} + \frac{\partial}{\partial x_j} v_i(\bar{x}(\alpha, t), t)\, \Delta t$

$$= \int_a^b \left\{ \left[f(\bar{x}(\alpha, t), t) + \frac{D}{Dt} f(\bar{x}(\alpha, t), t)\, \Delta t \right] \left[\delta_{ij} + \frac{\partial v_i(\bar{x}(\alpha, t), t)}{\partial x_j}\, \Delta t \right] - f(\bar{x}(\alpha, t), t)\, \delta_{ij} \right\} \frac{dx_j(\alpha, t)}{d\alpha}\, d\alpha =$$

$$= \int_a^b \left[\cancel{f(\bar{x}(\alpha, t), t)\, \delta_{ij}} + f(\bar{x}(\alpha, t), t)\, \frac{\partial v_i(\bar{x}(\alpha, t), t)}{\partial x_j}\, \Delta t + \frac{Df(\bar{x}(\alpha, t), t)}{Dt}\, \Delta t\, \delta_{ij} + \underbrace{\cancel{\frac{Df(\bar{x}(\alpha, t), t)}{Dt}\, \frac{\partial v_i}{\partial x_j}\, (\Delta t)^2}}_{\text{neglected since } \sim (\Delta t)^2} - \cancel{f(\bar{x}(\alpha, t), t)\, \delta_{ij}} \right] \cdot \frac{dx_j(\alpha, t)}{d\alpha}\, d\alpha =$$

$$= \int_a^b \Delta t \left[\frac{D}{Dt} f(\bar{x}(\alpha, t), t)\, \delta_{ij} + f(\bar{x}(\alpha, t), t)\, \frac{\partial v_i(\bar{x}(\alpha, t), t)}{\partial x_j} \right] \underbrace{\frac{dx_j(\alpha, t)}{d\alpha}\, d\alpha}_{= ds_j}$$

If we divide by Δt and take the limit $\Delta t \to 0$ we obtain the total derivative of the line integral:

$$\frac{D}{Dt} \int_{\gamma(t)} f(\bar{x}, t)\, ds_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{\gamma(t+\Delta t)} f(\bar{x}(t+\Delta t), t)\, ds_i - \int_{\gamma(t)} f(\bar{x}(t), t)\, ds_i \right] =$$

$$= \int_{\gamma(t)} \left[f(\bar{x}, t)\, \frac{\partial v_i}{\partial x_j} + \frac{Df(\bar{x}, t)}{Dt}\, \delta_{ij} \right] ds_j$$

where we have "reabsorbed" the parametrization

q.e.d.

⊙ Note that since $\partial v_i/\partial x_j \, ds_j = \partial_j v_i \, ds_j = dv_i$ we can also write the result as

$$\frac{D}{Dt}\int_{\gamma(t)} f(\bar{x}(t),t)\, ds_i = \underbrace{\int_{\gamma(t)} f(\bar{x}(t),t)\, dv_i}\, + \underbrace{\int_{\gamma(t)} \frac{D}{Dt} f(\bar{x}(t),t)\, ds_i}$$

where the two terms represent curve variation + f variation contributions respectively

⊙ Note that we can apply the result to vector functions $\bar{f}$ replacing the scalar $f \to f_i$ i-th component of the vector function, and then summing on the dummy index i:

$$\frac{D}{Dt}\int_{\gamma(t)} \bar{f}\cdot d\bar{x} = \int_{\gamma(t)} \left(f_i \partial_j v_i + \frac{D f_i}{Dt} \delta_{ij} \right) ds_j = \text{ either (a) or (b)}$$

(a) (since $f_i \delta_{ij} ds_j = f_i ds_i$)

$$= \int_{\gamma(t)} f_i\, dv_i + \int_{\gamma(t)} \frac{D}{Dt} f_i\, dx_i = \underline{\int_{\gamma(t)} \bar{f}\cdot d\bar{v} + \int_{\gamma(t)} \frac{D\bar{f}}{Dt}\cdot d\bar{x}}$$

(b) $= \int_{\gamma(t)} f_i \partial_j v_i \, ds_j + \int (\partial_t f_i + \underbrace{v_j \partial_j f_i}_{(\bar{v}\cdot\text{grad})})\, ds_i =$ swapping i and j in the first integral (it is fine and they are both dummy sums)

$$= \int_{\gamma(t)} (\partial_t f_i + f_j \partial_i v_j + v_j \partial_j f_i)\, ds_i =$$ (using $f_j \partial_i v_j = \partial_i (f_j v_j) - v_j \partial_i f_j$)

$$= \int_{\gamma(t)} [\partial_t f_i + \partial_i (f_j v_j) + \underbrace{v_j(\partial_j f_i - \partial_i f_j)}_{(*)}]\, ds_i =$$

$$= \int_{\gamma(t)} \{\partial_t f_i + \partial_i (f_j v_j) + [\mathrm{curl}(\bar{f}) \times \bar{v}]_i\}\, ds_i$$ or in vector form

$$\underline{\frac{D}{Dt}\int_{\gamma(t)} \bar{f}\cdot d\bar{x} = \int_{\gamma(t)} \left[\frac{\partial \bar{f}}{\partial t} + \mathrm{grad}(\bar{f}\cdot\bar{v}) + (\mathrm{curl}(\bar{f}) \times \bar{v})\right]\cdot d\bar{x}}$$

⊙ An important result comes out if $\bar{f} = \bar{v}$ velocity:

$$\frac{D}{Dt}\int_{\gamma(t)} \bar{v}\cdot d\bar{x} = \int_{\gamma(t)} \bar{v}\cdot d\bar{v} + \int_{\gamma(t)} \frac{D\bar{v}}{Dt}\cdot d\bar{x} = \int_{\gamma(t)} d\left(\tfrac{1}{2}v^2\right) + \int_{\gamma(t)} \frac{D\bar{v}}{Dt}\cdot d\bar{x}$$

exact differential

; if γ closed curve,

$$\boxed{\frac{D}{Dt}\oint_{\gamma(t)} \bar{v}\cdot d\bar{x} = \oint_{\gamma(t)} \frac{D\bar{v}}{Dt}\cdot d\bar{x}}$$

(*) Let us prove the equality. $[\mathrm{curl}(\bar{f}) \times \bar{v}]_i = \varepsilon_{ijk}[\varepsilon_{j\alpha\beta}\partial_\alpha f_\beta] v_k =$

$= \varepsilon_{kij}\varepsilon_{\alpha\beta j}\partial_\alpha f_\beta v_k = (\delta_{k\alpha}\delta_{i\beta} - \delta_{k\beta}\delta_{i\alpha})\partial_\alpha f_\beta v_k = (\partial_k f_i - \partial_i f_k) v_k$

(our statement, just call j the index k