Stability / Instability

When does a system, initially at rest or in a steady-state dynamics condition, go back to such state after being perturbed? And when does its state become irreversibly different?

A system is said to be stable or unstable, respectively, under the occurrence of the first or second instance. We shall consider this problem for fluids subjected to perturbations either of arbitrary or small amplitude, evaluating the different constraints arising in the two cases.

Arbitrary perturbations

Let us consider an incompressible viscous flow, whose steady-state, unperturbed solution yields a velocity field $\bar{v}_s$ and a pressure field p_s; the b.c. shall be $\bar{v}|_{S_R} = \bar{v}_0$ on the boundary S_R of the domain R. The perturbation, i.e. a time-dependent perturbation velocity field $\bar{u}(t)$, yields a perturbed solution $\bar{v} = \bar{v}_s + \bar{u}$, $p = p_s + p'$, with the same b.c. $\bar{v}|_{S_R} = \bar{v}_0$ (and both $\mathrm{div}\,\bar{v}_s = \phi$, $\mathrm{div}\,\bar{v} = \phi$).

The Navier-Stokes eq. for the perturbed and unperturbed flows reads, respectively

$$\frac{\partial \bar{u}}{\partial t} + \left[(\bar{v}_s + \bar{u}) \cdot \mathrm{grad} \right] (\bar{v}_s + \bar{u}) = -\frac{1}{\rho} \mathrm{grad}\,(p_s + p') + \nu \nabla^2 (\bar{v}_s + \bar{u})$$

$$(\bar{v}_s \cdot \mathrm{grad})\, \bar{v}_s = -\frac{1}{\rho} \mathrm{grad}\, p_s + \nu \nabla^2 \bar{v}_s$$

and subtracting the N.-S. eqs, the b.c. eqs. and incompressibility eqs. we get

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + (\bar{v}_s \cdot \mathrm{grad})\, \bar{u} + (\bar{u} \cdot \mathrm{grad})\, \bar{v}_s + (\bar{u} \cdot \mathrm{grad})\, \bar{u} = -\dfrac{1}{\rho} \mathrm{grad}\, p' + \nu \nabla^2 \bar{u} \quad (*) \\ \mathrm{div}\, \bar{u} = \phi \\ \bar{u}|_{S_R} = \phi \end{cases}$$

Now let us consider the kinetic energy of the perturbation, integrated over the whole domain R of the flow; if its time derivative is positive/negative, we can thus see whether the perturbation grows/dampens (i.e. the system is unstable/stable).

$$\frac{d}{dt} \left(\frac{1}{2} \int_R |\bar{u}^2|\, d^3x \right) = \int_R \bar{u} \cdot \frac{\partial \bar{u}}{\partial t}\, d^3x = \left(\text{using } (*) \right)$$

$$= \underbrace{-\int_R u_i (\partial_k u_i)\, v_k^s\, d^3x}_{①} \underbrace{- \int_R u_i (\partial_k v^s_i)\, u_k\, d^3x}_{②} \underbrace{- \int_R u_i (\partial_k u_i)\, u_k\, d^3x}_{③} \underbrace{- \frac{1}{\rho} \int_R u_i \partial_i p'\, d^3x}_{④} + \underbrace{\nu \int_R u_i \nabla^2 u_i\, d^3x}_{⑤}$$

Let us evaluate the terms one by one.

① Since $\mathrm{div}(|\bar{u}|^2 \bar{v}_s) = 2u_i(\partial_k u_i)v^s_k + |\bar{u}|^2 \underbrace{\partial_i v^s_i}_{\mathrm{div}\,\bar{v}_s = \phi} = 2u_i(\partial_k u_i)v^s_k$

$$\Rightarrow \int_R u_i(\partial_k u_i)v^s_k d^3x = \frac{1}{2}\int_R \mathrm{div}(|\bar{u}|^2\bar{v}_s)d^3x = \frac{1}{2}\int_{S_R} |\bar{u}|^2 \bar{v}_s \cdot \hat{n}\, da = \phi \qquad \text{since } \bar{u}\big|_{S_R} = \phi$$

③ This term is formally identical to ①:

$$\mathrm{div}(|\bar{u}|^2\bar{u}) = 2u_i(\partial_k u_i)u_k + |\bar{u}|^2 \underbrace{\partial_i u_i}_{\mathrm{div}\,\bar{u}=\phi} = 2u_i(\partial_k u_i)u_k$$

$$\Rightarrow \int_R u_i(\partial_k u_i)u_k d^3x = \frac{1}{2}\int_R \mathrm{div}(|\bar{u}|^2\bar{u})d^3x = \frac{1}{2}\int_{S_R} |\bar{u}|^2\bar{u}\cdot\hat{n}\, da = \phi \qquad \text{since } \bar{u}\big|_{S_R} = \phi$$

④ Quite similarly, $\mathrm{div}(p'\bar{u}) = \bar{u}\cdot\partial_i p' + p' \underbrace{\partial_i u_i}_{\phi} = u_i\partial_i p'$

$$\Rightarrow \int_R u_i\partial_i p' d^3x = \int_R \mathrm{div}(p'\bar{u})d^3x = \int_{S_R} p'\bar{u}\cdot\hat{n}\, da = \phi \qquad \text{since } \bar{u}\big|_{S_R} = \phi$$

② Here we notice that in $\partial_k v^s_i$ the contribution comes from the symmetric part only of the tensor,

$$\Rightarrow -\int_R u_i(\partial_k v^s_i)u_k d^3x = -\int_R u_i \frac{1}{2}(\partial_k v^s_i + \partial_i v^s_k)u_k d^3x$$

⑤ Notice that $\mathrm{div}(\mathrm{grad}|\bar{u}|^2) = \partial_k(2u_i\partial_k u_i) = 2(\partial_k u_i)(\partial_k u_i) + 2u_i\nabla^2 u_i$

and also that

$$\int_R \mathrm{div}(\mathrm{grad}|\bar{u}|^2)d^3x = \int_{S_R}(\mathrm{grad}|\bar{u}|^2)\cdot\hat{n}\, da = 2\int_{S_R} u_i\partial_k u_i n_k da = \phi \qquad \text{again as } \bar{u}\big|_{S_R} = \phi$$

$$\Rightarrow \nu\int_R u_i\nabla^2 u_i d^3x = -\nu\int_R(\partial_k u_i)(\partial_k u_i)d^3x$$

After all of this manipulation, we can summarize

$$\frac{d}{dt}\left(\frac{1}{2}\int_R |\bar{u}|^2 d^3x\right) = -\int_R u_i \frac{1}{2}(\partial_k v^s_i + \partial_i v^s_k)u_k d^3x - \nu\int_R(\partial_k u_i)(\partial_k u_i)d^3x$$

Notice that the second integral contains a squared quantity so that it is surely $\geq \phi$; moreover, it vanishes only when $\bar{u} = \phi$, so it is mathematically equivalent to <u>the square of a norm $\|\bar{u}\|$.</u>

The first integral has a mathematical representation as the average value of a symmetrical tensor operator $\underline{\underline{A}}$ with $A_{ik} \doteq -\frac{1}{2}(\partial_k v^s_i + \partial_i v^s_k)$, hence the average value

$$(\bar{u}, \underline{\underline{A}}\bar{u}) = -\frac{1}{2}\int_R u_i A_{ik} u_k d^3x$$

and with the said norm $\|\bar{u}\| \doteq \int_R (\partial_n u_i)(\partial_n u_i)\, d^3x$ we can write

$$\frac{d}{dt}(E_{K,pert}) = \frac{d}{dt}\left[\frac{1}{2}\int_R |\bar{u}|^2 d^3x\right] = (\bar{u}, \underline{\underline{A}}\bar{u}) - \nu \|\bar{u}\|^2$$

We can write the eigenvalue equation for the operator $\underline{\underline{A}}$: $\underline{\underline{A}}\bar{u} = \lambda\bar{u}$ $\quad (\Rightarrow (\bar{u}, \underline{\underline{A}}\bar{u}) = \lambda(\bar{u},\bar{u}))$ with eigenvalues λ_e; if $\nu > \max(\lambda_e)$, we can guarantee the stability condition $\frac{d}{dt}(E_{K,pert}) < \phi$, the kinetic energy of the perturbation is dissipated and indeed this is thanks to the viscosity of the flow.

Considering dimensionless quantities, since $Re = ul/\nu$ stability for large ν is equivalent to requesting relatively low values of Re. An explicit stability evaluation for the plane Poiseuille flow, for instance, would require $Re < 66$ (we shall not prove it here); yet many experimental observations show stability for much higher Re ($10^2 - 10^3$) – why? Because stability with respect to arbitrary perturbation amplitude sets a very tight, limiting constraint, while small perturbations relax such constraints and a system may still be stable with respect to small disturbances.

Small perturbations

When small (infinitesimal) perturbations are considered, we take another (general) description. As $\bar{u}$ (perturbation) is infinitesimal, the nonlinear term $(\bar{u}\cdot \mathrm{grad})\bar{u}$ is negligible and the Navier-Stokes eq. (*) for the perturbation is reduced to

$$\frac{\partial \bar{u}}{\partial t} = -(\bar{v}_s \cdot \mathrm{grad})\,\bar{u} - (\bar{u}\cdot \mathrm{grad})\,\bar{v}_s - \frac{1}{\rho}\,\mathrm{grad}\, p' + \nu\nabla^2\bar{u}$$

i.e. a linear differential eq. in $\bar{u}$: $\frac{\partial \bar{u}}{\partial t} = \underline{\underline{L}}\bar{u}$ with $\underline{\underline{L}}$ linear differential operator

and we may consider the solutions to the eigenvalue equation for $\underline{\underline{L}}$

$\underline{\underline{L}}\bar{u} = \lambda\bar{u}$ with $\lambda \in \mathbb{C}$ complex eigenvalues $\lambda \doteq \gamma + i\omega$, the solutions are

$$\bar{u}(t) = \bar{u}_0 \exp[(\gamma + i\omega)t]$$

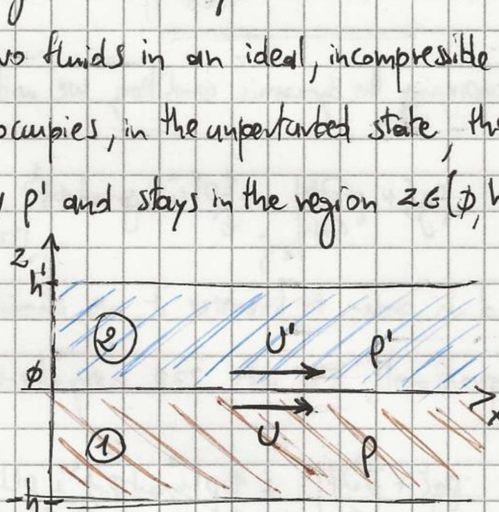
where it is apparent that the real part γ of the eigenvalue λ determines the stability

$$\gamma = \mathrm{Re}(\lambda) \begin{cases} > \phi & \text{unstable} \\ < \phi & \text{stable} \end{cases}$$

Instability of tangential discontinuities

Let us revisit the system made out of two adjacent fluid layers – which we discussed when dealing with gravity waves – and consider the two fluids in an ideal, incompressible potential flow. The bottom layer has density ρ and occupies, in the unperturbed state, the region $z \in [-h, 0]$, while the upper layer has density ρ' and stays in the region $z \in [0, h']$.

We now evaluate the chances of instability when the fluids are not at rest, but their steady state is a flow with fundamental velocities respectively



$$\begin{cases} \vec{u} = U\hat{e}_x & (1) \\ \vec{u}' = U'\hat{e}_x & (2) \end{cases}$$

i.e. there is a relative sliding motion between the two fluids. The solution, which we assume to be y-invariant like the problem, is given by two fluid potential satisfying the Laplace eq.

$\nabla^2\varphi = 0$, $\nabla^2\varphi' = 0$ and for simplicity we treat h, h' as very large ($h, h' \to \infty$), thus requesting φ, φ' to stay finite for $z \to \mp\infty$, respectively. The fundamental (unperturbed) potentials are $\varphi_0(x) = Ux$,

$\varphi_0'(x) = U'x$;

we consider ∞esimal perturbations ϕ, ϕ' to φ_0, φ_0' respectively so that the perturbed potentials are

$$\varphi(x,z,t) = \varphi_0(x) + \phi(x,z,t) = Ux + \phi(x,z,t)$$

$$\varphi'(x,z,t) = \varphi_0'(x) + \phi'(x,z,t) = U'x + \phi'(x,z,t)$$

The kinematic condition at the interface (continuity of velocity = no detachment) for explicit form of the interface reads

$$\begin{cases} \left.\dfrac{\partial\zeta}{\partial t} + \dfrac{\partial\zeta}{\partial x} v_x\right|_{z=\zeta} - v_z(x,\zeta,t) = 0 \\ \left.\dfrac{\partial\zeta}{\partial t} + \dfrac{\partial\zeta}{\partial x} v_x'\right|_{z=\zeta} - v_z'(x,\zeta,t) = 0 \end{cases}$$

Notice that $v_x = \partial_x\varphi = U + \partial_x\phi$

$v_x' = \partial_x\varphi' = U' + \partial_x\phi'$

where U, U' are finite terms while $\partial_x\phi, \partial_x\phi'$ are infinitesimal and can be neglected, when we linearize:

$$\boxed{\begin{cases} \partial_t \zeta + U \partial_x \zeta = \partial_z \phi \big|_{z=0} \\ \partial_t \zeta + U' \partial_x \zeta = \partial_z \phi' \big|_{z=0} \end{cases}}$$ <u>linearized kinematic conditions</u>

Concerning the dynamic condition, we write the generalized Bernoulli eq.

$$\rho g \zeta + \rho \frac{\partial \phi}{\partial t}\Big|_{z=\zeta} + \frac{1}{2}\rho \left|\bar{u} + \text{grad}\,\phi\right|^2\Big|_{z=\zeta} = \rho' g \zeta + \frac{\partial \phi'}{\partial t}\Big|_{z=\zeta} + \frac{1}{2}\rho' \left|\bar{u} + \text{grad}\,\phi'\right|^2\Big|_{z=\zeta}$$

and in order to linearize it we eliminate the quadratic infinitesimal terms like $|\text{grad}\,\phi|^2$ and approximate $z=\zeta$ with $z=0$ everywhere, with the exception of the terms $\rho g \zeta$, $\rho' g \zeta$, thus obtaining

$$\rho g \zeta + \rho \frac{\partial \phi}{\partial t}\Big|_{z=0} + \frac{1}{2}\rho U^2 + \rho U \frac{\partial \phi}{\partial x}\Big|_{z=0} = \rho' g \zeta + \rho' \frac{\partial \phi'}{\partial t}\Big|_{z=0} + \frac{1}{2}\rho' U'^2 + \rho' U' \frac{\partial \phi'}{\partial x}\Big|_{z=0} \qquad (*)$$

while the condition for the unperturbed flows is

$$\rho g \zeta + \rho \frac{\partial \varphi_0}{\partial t}\Big|_{z=0} + \frac{1}{2}\rho |\text{grad}\,\varphi_0|^2 = \rho' g \zeta + \rho' \frac{\partial \varphi_0'}{\partial t}\Big|_{z=0} + \frac{1}{2}\rho' |\text{grad}\,\varphi_0'|^2$$

but in the unperturbed situation $\zeta = 0$, $\partial_t \varphi_0 = \partial_t \varphi_0' = 0$, $|\text{grad}\,\varphi_0|^2 = U^2$, $|\text{grad}\,\varphi_0'|^2 = U'^2 \Rightarrow$

$\frac{1}{2}\rho U^2 = \frac{1}{2}\rho' U'^2$ which can be replaced in the perturbed i.b.c. $(*)$ yielding

$$\boxed{\rho g \zeta + \rho \frac{\partial \phi}{\partial t}\Big|_{z=0} + \rho U \frac{\partial \phi}{\partial x}\Big|_{z=0} = \rho' g \zeta + \rho' \frac{\partial \phi}{\partial t}\Big|_{z=0} + \rho U \frac{\partial \phi}{\partial x}\Big|_{z=0}}$$ <u>linearized dynamic condition</u>

Now we specify the fact that the perturbations are in the form of <u>a monochromatic wave</u>:

$\phi(x,z,t) = f(z) \exp[i(kx - \omega t)]$

$\phi'(x,z,t) = f'(z) \exp[i(kx - \omega t)]$

and plug them into the Laplace eq.; hence we get for ϕ

$$\frac{\partial^2 f(z)}{\partial z^2} \exp[i(kx-\omega t)] - k^2 f(z) \exp[i(kx - \omega t)] = 0 \quad \rightsquigarrow$$

$$\frac{\partial^2 f(z)}{\partial z^2} = k^2 f(z) \qquad \text{and similarly for } \phi' \quad \frac{\partial^2 f'(z)}{\partial z^2} = k^2 f'(z);$$

as $\phi(z \to -\infty)$ and $\phi'(z \to +\infty)$ must remain finite, the solutions must be

$f(z) = A \exp(kz)$

$f'(z) = B \exp(-kz)$ and we can now plug them into the dynamic i.b.c.:

$$\rho g \zeta - i\omega\rho A \exp(0) \exp[i(kx - \omega t)] + ik\rho U A \exp(0) \exp[i(kx-\omega t)] =$$

$$= \rho' g \zeta - i\omega\rho' B \exp(0) \exp[i(kx - \omega t)] + ik\rho' U' B \exp[i(kx - \omega t)]$$

which we can invert to isolate ζ:

$$\zeta(x,t) = \frac{1}{g(\rho-\rho')}\left[i(U'k-\omega)\rho' B - i(Uk-\omega)\rho A\right]\exp[i(kx-\omega t)],$$

inserting $\zeta(x,t)$ into the kinematic i.b.c. with some honestly tedious algebra we obtain

$$\begin{cases} A\left[\omega^2\rho - 2k\omega\rho U + k^2\rho U^2 - g(\rho-\rho')k\right] - B\left[-\omega^2\rho' + 2k\omega\rho'(U+U') - k^2\rho' UU'\right] = \phi \\ A\left[\omega^2\rho - k\omega\rho(U+U') + k^2\rho UU'\right] - B\left[-\omega^2\rho' + 2k\omega\rho' U' - k^2\rho' U'^2 + g(\rho-\rho')k\right] = \phi \end{cases}$$

i.e. a homogeneous linear system with non-trivial solutions found by setting $\det \underline{\underline{M}} = \phi$ (where we call $\underline{\underline{M}}$ the matrix of the coefficients of A and B); $\det\underline{\underline{M}}=\phi$ yields the dispersion relation

$$\boxed{\frac{\omega}{k} = \frac{\rho U+\rho' U'}{\rho+\rho'} \pm \left[\frac{g(\rho-\rho')}{k(\rho+\rho')} - \frac{\rho\rho'}{(\rho+\rho')^2}(U-U')^2\right]^{1/2}} \quad \left(= \frac{\rho U+\rho' U'}{\rho+\rho'} \pm (\alpha-\beta)^{1/2}\right)$$

and we can make a number of remarks.

① If $\alpha > \beta$, we have neutral stability (ω/k is real and waves are stable, i.e. neither grow nor decay, once they are started).

② We ~~have~~ have both solutions $\pm(\alpha-\beta)^{1/2}$, therefore if $\alpha<\beta$ we have an imaginary root and the solution with $+(\alpha-\beta)^{1/2}$, when plugged into $e^{-i\omega t}$, yields a real exponential with positive exponent, i.e. a growing perturbation $\Rightarrow$ instability.

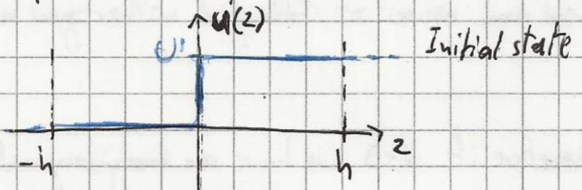
③ If $U=U'=\phi$ we recover the dispersion relation already found for gravity waves at an interface between fluids, $\omega^2 = gk(\rho-\rho')/(\rho+\rho')$.
Notice that if $\rho'>\rho$, i.e. the upper fluid is denser (heavier), $\alpha<\phi \Rightarrow \pm\sqrt{\alpha}$ is imaginary and again we obtain an instability called <u>Rayleigh-Taylor instability</u>.

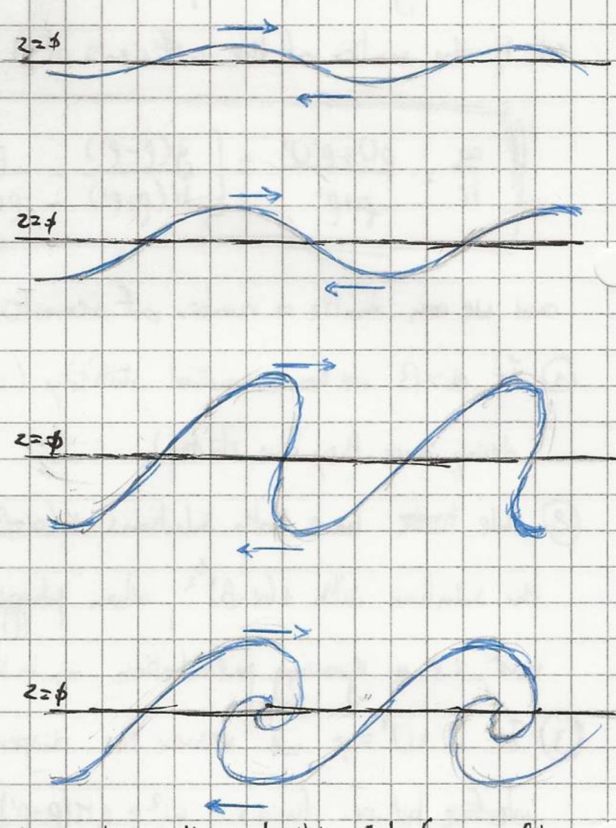
④ For $U\neq U'\neq\phi$, a negative $(\alpha-\beta)$ term can always be obtained, provided there is a sufficiently large k (small λ = short waves):
<u>The flow is always unstable for short waves</u>, as soon as there is a velocity difference $U-U'\neq\phi$.

⑤ $\alpha=\phi$, i.e. $g=\phi$ or $\rho=\rho'$ and $U\neq U'$ $\Rightarrow$ <u>Kelvin-Helmholtz instability</u>
In particular $\rho=\rho'$ yields $\boxed{\frac{\omega}{k} = \frac{1}{2}(U+U') \pm i\frac{1}{2}(U-U')}$

and this is the case where there is just one fluid ($\rho=\rho'$) but layered with a <u>velocity discontinuity</u> (see VORTEX SHEET). This instability occurs at all wavelengths, and the unstable wave travels with a phase velocity $v_{ph} = \mathrm{Re}\left(\frac{\omega}{k}\right) = \frac{1}{2}(U+U')$ average speed of the fundamental (unperturbed) flow, and from a CS moving with v_{ph} we can see a symmetric situation of the two layers flowing with equal and opposite fundamental velocity $\pm\frac{1}{2}(U-U')$ (no preferential propagation orientation for the wave).

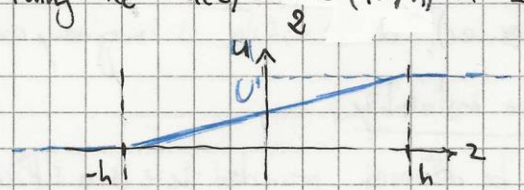
The Kelvin-Helmholtz instability is generated by the destabilizing effect of the velocity shear that overcomes the stabilizing effect of stratification; the energy source for the instability is the kinetic energy of the sheared flow, and the perturbations work by "smearing" the velocity gradients until they are ultimately damped. Intuitively, let us imagine the initial velocity profile is step-like: $U=0$, $U'\neq 0$ over a layer where the perturbation takes place $z \in [-h, h]$.





Kelvin-Helmholtz instability: Interface profile

After the instability has grown and mixed the layers at the interface (see figure on the right), the velocity profile $u(z)$ will be smoothed out into something like $u(z) = \frac{1}{2}U'(1+z/h)$ in $z \in [-h, h]$:



The kinetic energy per unit cross-sectional area is $E_k = \int_{-h}^{h} \frac{1}{2}\rho u^2(z)\,dz$; the initial value is

$$E_{k_i} = \frac{1}{2}\rho U'^2 h$$

while the final one is

$$E_{k_f} = \frac{1}{4}\rho U' \int_{-h}^{h} (1+z/h)^2\,dz = \frac{\rho}{3}U'^2 h < E_{k_i}$$ showing that by attenuating the velocity (shear) gradient, the perturbation energy has decreased (while momentum is conserved, $\int_{-h}^{h} u(z)\,dz$ = constant).