

Heat transport

Since the beginning we have noticed that the description of a fluid's dynamics requires five scalar equations - a number equal to the number of scalar quantities that define the state of the system (\bar{v}, p, ρ) . In the framework of ideal fluids we identified the equations as Euler's equation, the continuity equation and the adiabatic equation.

Moving to real - viscous - fluids we must replace Euler's eq. with the Navier-Stokes eq., but there is more than that: The dissipation introduced by viscosity (and hence irreversibility) require us to revisit the conservation of entropy (adiabatic eq.) as it does no longer hold - in other words, we must consider heat exchange.

Heat exchange in the energy balance equation

Let us calculate the power supplied by surface forces to a continuum element - and we may keep it a generic continuous medium, up to the point where we explicitly use constitutive laws for the material and, as we will do later, Fourier's law of heat conduction.

The convention we shall use considers as positive the work absorbed and the heat released by the element (quite the opposite of frequent thermodynamics conventions...):

If the element occupies a region R with boundary S_R , the remaining continuum exerts a force through the surface of infinitesimal area da at position \bar{x} and normal unit vector \hat{n} (directed outwards) which we express in components as

$$d\vec{F}_i(\bar{x}) = \sigma_{ij}(\bar{x}) n_j(\bar{x}) da \quad \text{supplying a power}$$

$$dP(\bar{x}) = d\vec{F}_i(\bar{x}) v_i(\bar{x}) = \sigma_{ij}(\bar{x}) v_i(\bar{x}) n_j(\bar{x}) da \quad \text{and by integration over the whole } S_R$$

$$P = \int_{S_R} \sigma_{ij}(\bar{x}) v_i(\bar{x}) n_j(\bar{x}) da = \int_R \partial_j (\sigma_{ij} v_i) d^3x \quad * \quad \text{and approximating the integral for mesimal } R,$$

with the divergence theorem

$$\dot{w} = P/M = \frac{1}{\rho} \partial_j (\sigma_{ij} v_i) \quad \text{power per unit mass}$$

* = Worth saying again: A continuum element is not a point particle. Therefore

$$P = \sum_r \vec{F}_r^{\text{ext}} \cdot \vec{v}_r = \sum_r \vec{F}_r^{\text{ext}} \cdot (\vec{v}_r - \vec{v}_{cm}) + \sum_r \vec{F}_r^{\text{ext}} \cdot \vec{v}_{cm} = \sum_r \vec{F}_r^{\text{ext}} \cdot (\vec{v}_r - \vec{v}_{cm}) + \vec{R} \cdot \vec{v}_{cm}$$

(sum over the microscopic constituents and forces acting on each)

\Rightarrow power P is not only given by the resultant of forces on the center of mass; therefore we calculate P as the integral above

What is this power used for? It is stored in mechanical and internal energy. The rate of change of mechanical energy per unit mass is

$$\frac{D}{Dt}(E_{\text{mech}}) = \frac{D}{Dt}\left(\frac{1}{2}v^2 + u\right) = \bar{v} \cdot \frac{D\bar{v}}{Dt} + \frac{Du}{Dt} = v_i \frac{1}{\rho} \partial_j \sigma_{ij} - v_i \partial_i u + v_i \partial_i u = \frac{1}{\rho} v_i \partial_j \sigma_{ij} =$$

$$\frac{Dv_i}{Dt} = \frac{1}{\rho} \partial_j \sigma_{ij} - \partial_i u \quad \left(\frac{1}{\rho} \partial_j \sigma_{ij} + v_i \partial_i u \right) = \frac{1}{\rho} \partial_j (v_i \sigma_{ij}) - \frac{1}{\rho} \sigma_{ij} \partial_j v_i$$

Since $\dot{w} = \frac{D}{Dt}(E_{\text{mech}} + E) = \frac{1}{\rho} \partial_j (v_i \sigma_{ij}) \Rightarrow \frac{DE}{Dt} = \frac{1}{\rho} \sigma_{ij} \partial_j v_i$ by subtraction

For Newtonian fluids we could specify

$$\frac{1}{\rho} \sigma_{ij} \partial_j v_i = \frac{1}{\rho} \sigma_{ij} \bar{\Pi}_{ij} = \frac{1}{\rho} \sigma_{ij}^S \bar{\Pi}_{ij}^S + \frac{1}{\rho} \sigma_{ij}^T \bar{\Pi}_{ij}^T =$$

$$= \frac{\eta}{\rho} \sum_{ij} (\partial_j v_i + \partial_i v_j - \frac{2}{3} \text{div}(\bar{v}) \delta_{ij})^2 - \frac{\zeta}{\rho} (\text{div}(\bar{v}))^2 - \frac{1}{\rho} p \text{div}(\bar{v})$$

L

Now we consider an out-of-equilibrium situation, i.e. we include the possibility of heat exchange, so that $\frac{D}{Dt}\left(\frac{1}{2}v^2 + u + E\right) = \dot{w} - \dot{q}$ where \dot{q} is the heat per unit mass and time released by the element of continuum (and this is basically the first law of thermodynamics)

and the heat exchange can be expressed as the occurrence of a heat flux \bar{q} such that

$$\dot{q} = \frac{1}{\rho} \text{div} \bar{q} \quad (\text{continuity equation for heat}) \quad \text{hence}$$

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + u + E\right) = \dot{w} - \dot{q} = \frac{1}{\rho} \partial_j (v_i \sigma_{ij}) - \frac{1}{\rho} \text{div} \bar{q}$$

and again, since $\frac{D}{Dt}\left(\frac{1}{2}v^2 + u\right) = \frac{1}{\rho} v_i \partial_j \sigma_{ij}$, by subtraction we get

$$\boxed{\frac{DE}{Dt} = \frac{1}{\rho} \sigma_{ij} \partial_j v_i - \frac{1}{\rho} \text{div} \bar{q}}$$

equation that expresses the heat exchange in an energy balance form per unit mass

Using the expression $\frac{D}{Dt}(p\bar{v}) + \text{div}(p\bar{q}) = p \frac{D\bar{v}}{Dt}$ we also have

$$\frac{D}{Dt}(pE) = -\text{div}(pE\bar{v}) + p \frac{DE}{Dt} \Rightarrow \boxed{\frac{D}{Dt}(pE) = -\text{div}(pE\bar{v}) + \sigma_{ij} \partial_j v_i - \text{div} \bar{q}}$$

the equivalent

heat equation as an energy balance per unit volume

and

$$\frac{D}{Dt}[p(E_{\text{mech}} + E)] = -\text{div}[p(E_{\text{mech}} + E)\bar{v}] + p \frac{D}{Dt}(E_{\text{mech}} + E) =$$

$$\begin{aligned}
 &= -\operatorname{div}[\rho(E_{\text{mech}} + E)\bar{v}] + \partial_j(v_i \sigma_{ij}) - \operatorname{div} \bar{q} = \quad \text{with } \sigma_{ij} = -p\delta_{ij} + \sigma'_{ij} \\
 &= -\operatorname{div}[\rho(E_{\text{mech}} + E)\bar{v}] - \partial_j(p\delta_{ij} v_i) + \partial_j(\sigma'_{ij} v_i) - \operatorname{div} \bar{q} = \\
 &= -\operatorname{div}[\rho(E_{\text{mech}} + E)\bar{v}] - \partial_i(p v_i) + \partial_j(\sigma'_{ij} v_i) - \operatorname{div} \bar{q} = \\
 &= -\operatorname{div}[\rho(E_{\text{mech}} + \underbrace{E + p/\rho}_w)\bar{v}] + \partial_j(\sigma'_{ij} v_i) - \operatorname{div} \bar{q}
 \end{aligned}$$

w enthalpy

$$\Rightarrow \frac{\partial}{\partial t} [\rho(E_{\text{mech}} + E)] = \underbrace{\partial_j(\sigma'_{ij} v_i)}_{\substack{\text{work done} \\ \text{by viscous} \\ \text{surface forces} \\ \text{per unit time}}} - \underbrace{\operatorname{div} \bar{q}}_{\substack{\text{heat} \\ \text{flux}}} - \underbrace{\operatorname{div}[\rho(E_{\text{mech}} + w)\bar{v}]}_{\substack{\text{advective total} \\ \text{energy flux}}}$$

i.e. rate of change of total energy per unit volume

Which we can also integrate over the fixed region R , making the interpretation even clearer:

$$\frac{d}{dt} (\bar{E}_{\text{mech}} + \bar{E}_{\text{int}}) = \frac{d}{dt} \int_R \rho(E_{\text{mech}} + E) d^3x = \int_R \frac{\partial}{\partial t} [\rho(E_{\text{mech}} + E)] d^3x =$$

$$= - \int_R \operatorname{div} \bar{q} d^3x - \int_R \operatorname{div} [\rho(E_{\text{mech}} + w)\bar{v}] d^3x + \int_R \partial_j(\sigma'_{ij} v_i) d^3x = \quad \text{with the divergence theorem}$$

$$= - \int_{S_R} \bar{q} \cdot d\bar{a} - \int_{S_R} \rho(E_{\text{mech}} + w)\bar{v} \cdot d\bar{a} + \int_{S_R} \sigma'_{ij} v_i n_j d\bar{a}$$

Notice here that if R coincides with the domain and thus $\bar{v}|_{S_R} = \bar{v}$, only heat exchange can contribute to energy variation and in an isolated system (no heat exchange with the outside world, $\bar{q}|_{S_R} = \bar{0}$) the total energy is conserved.

Heat transfer equation in terms of entropy balance

The differential of the internal energy per unit mass of a continuum element is

$$dE = T ds - p d(1/\rho) = T ds - p d(1/\rho) = T ds + \frac{p}{\rho^2} d\rho$$

and the expression can be equivalently written in terms of total derivatives just dividing by dt:

$$\frac{DE}{Dt} = T \frac{Ds}{Dt} - p \frac{D(1/\rho)}{Dt} = T \frac{Ds}{Dt} - \frac{p}{\rho} \operatorname{div} \bar{v} \quad (\text{using continuity: } \frac{1}{\rho} \frac{D(1/\rho)}{Dt} = \operatorname{div} \bar{v})$$

Plugging into this expression the result we just found $\frac{DE}{Dt} = \frac{1}{\rho} \sigma_{ij} \partial_j v_i - \frac{1}{\rho} \operatorname{div} \bar{q}$, we get

$$\begin{aligned} T \frac{Ds}{Dt} &= \frac{1}{\rho} \sigma_{ij} \partial_j v_i - \frac{1}{\rho} \operatorname{div} \bar{q} + \frac{p}{\rho} \operatorname{div} \bar{v} = \\ &= \frac{1}{\rho} \underbrace{(-p \delta_{ij} + \sigma_{ij})}_{\substack{\hookrightarrow -\frac{p}{\rho} \delta_{ij} \partial_j v_i = -\frac{p}{\rho} \operatorname{div} \bar{v} \\ \hookrightarrow -\frac{p}{\rho} \partial_j v_i = -\frac{p}{\rho} \operatorname{div} \bar{v}}} \partial_j v_i - \frac{1}{\rho} \operatorname{div} \bar{q} + \frac{p}{\rho} \operatorname{div} \bar{v} = \frac{1}{\rho} \sigma'_{ij} \partial_j v_i - \frac{1}{\rho} \operatorname{div} \bar{q} \end{aligned}$$

$$\Rightarrow \left| \frac{Ds}{Dt} = \frac{1}{\rho T} \sigma'_{ij} \partial_j v_i - \frac{1}{\rho T} \operatorname{div} \bar{q} \right| \quad \begin{array}{l} \text{general eq. of heat transfer} \\ \text{(entropy balance eq. per unit mass)} \end{array}$$

Since $\operatorname{div}(\bar{q}/T) = \frac{1}{T} \operatorname{div} \bar{q} + \bar{q} \cdot \operatorname{grad}(1/T) = \frac{1}{T} \operatorname{div} \bar{q} - \frac{\bar{q} \cdot \operatorname{grad} T}{T^2}$, we can also rewrite it as

$$\Rightarrow \left| \frac{Ds}{Dt} = \underbrace{-\frac{1}{\rho} \operatorname{div} \left(\frac{\bar{q}}{T} \right)}_{\substack{\text{the term usually written} \\ \text{as } dQ/T \text{ in the} \\ \text{second law of thermodynamics,} \\ \text{yielding the entropy rate of} \\ \text{change in a reversible process}}} + \underbrace{\frac{1}{\rho T} \sigma'_{ij} \partial_j v_i}_{\substack{\text{mechanical energy} \\ \text{converted in internal energy,} \\ \text{divided by } T. \text{ structure} \\ \text{similar to the first term,} \\ \text{same interpretation as } dQ/T; \\ \text{here there is heat generated} \\ \text{by friction forces}}} - \underbrace{\frac{1}{\rho} \frac{\bar{q} \cdot \operatorname{grad} T}{T^2}}_{\substack{\text{term contributing to entropy} \\ \text{variation that comes directly} \\ \text{from the presence of heat} \\ \text{conduction in turn generated} \\ \text{by a temperature gradient}}}$$

Once again we rewrite the equation as expression per unit volume:

$$\begin{aligned} \frac{\partial(\rho s)}{\partial t} &= -\operatorname{div}(\rho s \bar{v}) + \rho \frac{Ds}{Dt} = -\operatorname{div}(\rho s \bar{v}) + \frac{1}{T} \sigma'_{ij} \partial_j v_i - \operatorname{div} \left(\frac{\bar{q}}{T} \right) - \frac{\bar{q} \cdot \operatorname{grad} T}{T^2} = \\ &= -\operatorname{div} \left(\rho s \bar{v} + \frac{\bar{q}}{T} \right) + \frac{1}{T} \sigma'_{ij} \partial_j v_i - \frac{\bar{q} \cdot \operatorname{grad} T}{T^2} \end{aligned}$$

and by integration over a fixed region R , and using the divergence theorem, yields yet another form of the second law of thermodynamics, where we express the fact that entropy cannot decrease in an isolated system:

$$\frac{dS}{dt} = \int_R \frac{\partial}{\partial t} (\rho s) d^3x = \underbrace{- \int_{S_R} \rho s \vec{u} \cdot \vec{n} da}_{(1)} - \underbrace{\int_{S_R} \frac{1}{T} \vec{q} \cdot \vec{n} da}_{(2)} + \underbrace{\int_R \frac{1}{T} \sigma_{ij}' \partial_j v_i d^3x}_{(3)} - \underbrace{\int_R \frac{1}{T^2} (\vec{q} \cdot \text{grad } T) d^3x}_{(4)}$$

(1) is an entropy flux term (zero if $R =$ entire domain)

(2) heat exchange term (zero if $R =$ entire domain and isolated system)

(3) + (4) are ENTROPY PRODUCTION TERMS within the volume; even if (1) and (2) vanish, entropy is not conserved because of these production terms, and these must be positive - this is

a way to better express the Clausius inequality $dS > \delta Q/T$, we know that entropy increases because of these production mechanisms. Notice that since (3) and (4) describe separate, independent phenomena we must require both of them to be separately positive, not only their sum. This sets some specific constraints on these integrals.

* $\sigma_{ij}' \partial_j v_i \geq \phi$; using the decomposition of σ_{ij}' and $\partial_j v_i = \dot{\Pi}_{ij}$,

$$\begin{aligned} \sigma_{ij}' \partial_j v_i &= \sigma_{ij}' \dot{\Pi}_{ij} = \sigma_{ij}'^S \dot{\Pi}_{ij}^S + \sigma_{ij}'^A \dot{\Pi}_{ij}^A = \\ &= \eta_r \left(\partial_j v_i + \partial_i v_j - \frac{2}{3} \text{div}(\vec{v}) \delta_{ij} \right)^2 + \zeta \left(\text{div}(\vec{v}) \right)^2 \geq \phi \end{aligned}$$

(we consider zero the antisymmetric part that is ok e.g., for a Newtonian fluid)

and since the traceless symmetric and isotropic components are independent and must be $\geq \phi$ separately, $\Rightarrow \boxed{\eta, \zeta \geq \phi}$; The viscosity coefficients are necessarily positive in order to comply with the constraints of thermodynamics.

* $(\vec{q} \cdot \text{grad } T) \leq \phi$

Thermodynamics tells us that in non-equilibrium processes flux densities of extensive quantities (like heat) depend on the gradients of intensive quantities and these relations are expressed by phenomenological laws. When we consider the heat flux density \vec{q} , we can in principle relate it to the gradients of the two thermodynamic variables p and T :

$$\vec{q} = \vec{q}(g \text{rad } p, g \text{rad } T)$$

More explicitly, we can write a phenomenological law where \vec{q} is expressed as a sum of multilinear terms in the gradients of p and T up to an order N - and then we shall see what kind of constraints are set on the terms of this sum by the request $(\vec{q} \cdot \text{grad } T) \leq \phi$.

We write \vec{q} in components q_i as

$$q_i = - \left(A_i^{(\phi, \phi)} + A_{ij}^{(\phi, 1)} \partial_j T + A_{ij}^{(\phi, 1)} \partial_j p + \right. \\
+ A_{ijk}^{(\phi, 2)} \partial_j T \partial_k T + A_{ijk}^{(\phi, 2)} \partial_j T \partial_k p + A_{ijk}^{(\phi, 2)} \partial_j p \partial_k p + \\
\left. + A_{ijkl}^{(\phi, 3)} \partial_j T \partial_k T \partial_l T + A_{ijkl}^{(\phi, 3)} \partial_j T \partial_k T \partial_l p + A_{ijkl}^{(\phi, 3)} \partial_j T \partial_k p \partial_l p + A_{ijkl}^{(\phi, 3)} \partial_j p \partial_k p \partial_l p + \dots \right)$$

where $A_{j_1 j_2 \dots j_r j_{r+1} \dots j_m}$ tensors of rank $m+1$ (with $m \in \mathbb{N}$) are the coefficients of the multilinear form of order r in $\text{grad} T$ and $m-r$ in $\text{grad} p$.

The series of terms is interrupted by establishing how small the gradients are in such a way to accept as negligible the terms of infinitesimal order higher than N .

The thermodynamic constraint $\bar{q} \cdot \text{grad} T \leq \phi$ must be evaluated separately, term by term.

① $A^{(\phi, \phi)} \neq \phi$ necessarily, otherwise we have a term $-A^{(\phi, \phi)} \text{grad} T$ that is not guaranteed to be negative (and also by common sense, $q_i \propto A_i^{(\phi, \phi)}$ would say that there can be a heat flux in the absence of temperature gradients).

② Likewise, since we multiply by $\text{grad} T$, terms with even factors of $\text{grad} T$ and odd factors in $\text{grad} p$ are not guaranteed to be negative and their coefficients must vanish.

③ As a consequence, only odd - grad T and even - grad p terms are acceptable, yielding a form

$$q_i = - \left(A_{ij}^{(\phi, \phi)} \partial_j T + A_{ijk}^{(\phi, 2)} \partial_j T \partial_k T + A_{ijk}^{(\phi, 2)} \partial_j T \partial_k p + \dots \right)$$

The simplest form is the linear expression, where only $A_{ij}^{(\phi, \phi)} \neq \phi$ and specifically it must be positive, so let us rename it $A_{ij}^{(\phi, \phi)} \doteq K_{ij} \geq \phi$ and

$$\underline{q_i = -K_{ij} \partial_j T}$$

even simpler, if we can assume an isotropic material the 2nd-order tensor K_{ij} is a scalar K thermal conductivity coefficient and

$$\boxed{\bar{q} = -K \text{grad} T} \quad \text{Fourier's law (law of heat conduction)}$$

An important note: Using Fourier's law, the second entropy production term reads

$$-\frac{1}{T^2} \bar{q} \cdot \text{grad} T = -\frac{1}{T^2} K |\text{grad} T|^2 \propto (\Delta T/T)^2 \quad \text{i.e. it is proportional to the square}$$

of the relative differences in temperature. Therefore, it is true that heat transfer is a source

of irreversibility, but this is a second-order effect in the temperature differences, and if we stop at the first order, heat transfer is a reversible process (see reversible heat exchange with thermostats as it is usually described in basic thermodynamics treatments).

Another significant note: In conclusion, the properties of a viscous and conductive fluid concerning entropy generation and thus irreversibility of its dynamics are contained in the parameters η , ζ and κ .

Heat equation for an incompressible fluid

We have often (well, always, when dealing with viscous fluid) taken advantage of the incompressibility property of flows where $v \ll c$ (subsonic flows) and pressure variations due to the motion of fluid elements result to be negligible - so that other thermodynamic variables, and specifically ρ density, experience negligible variations, too. Yet heating and thus temperature changes affect density; temperature gradients result in pressure gradients and an incompressibility assumption may not be entirely acceptable. In operational terms: Careful when dealing with derivatives of thermodynamic variables, as taking a derivative "at constant p " will be unwise and one should rather work at constant pressure.

In order to consider a flow as incompressible, small variations of p are requested
 a) under mechanical actions, b) under thermal stresses (thermal expansion).

Let us evaluate mechanical incompressibility and what kind of constraints it poses. We write the differential of the specific entropy s as

$$ds(p, T) = \frac{\partial s}{\partial T} dT + \frac{\partial s}{\partial p} dp = \frac{c_p}{T} dT - \frac{\beta}{\rho} dp;$$

↳ iv Maxwell relation ↗

What is the consequence of requiring the second term to be negligible (small variations, p constant)?

Comparing the two terms, if $|\Delta p| \ll \frac{\rho c_p |\Delta T|}{\beta T}$

$$\Rightarrow ds \approx \frac{\partial s}{\partial T} dT = \frac{c_p}{T} dT \quad \text{i.e. equivalently} \quad \frac{Ds}{DT} = \frac{c_p}{T} \frac{DT}{DT} (*)$$

Now when we write

$$\frac{1}{\rho} dp = \frac{1}{\rho} \frac{\partial \rho(p, s)}{\partial p} dp + \frac{1}{\rho} \frac{\partial \rho(p, s)}{\partial s} ds$$

mechanical incompressibility requires the first term (indeed, the mechanical compressibility term: ρ variation due to p variation) to be negligible with respect to the second one (indeed, describing thermal expansion). As we already know,

$$\frac{\partial \rho}{\partial p} = \frac{1}{c^2}; \quad \frac{1}{\rho} |\Delta p| \sim \frac{UL}{c^2}; \quad \frac{1}{\rho} \frac{\partial \rho}{\partial s} = -\frac{\beta T}{c_p}; \quad |\Delta s| \sim \frac{c_p}{T} |\Delta T|$$

therefore requesting $\frac{1}{\rho} \frac{\partial \rho}{\partial p} dp \ll \frac{1}{\rho} \frac{\partial \rho}{\partial s} ds$

is equivalent to

$$\frac{UL}{c^2} \frac{1}{c^2} \ll \frac{\beta T}{c_p} \frac{c_p}{T} |\Delta T| \Rightarrow \boxed{\frac{UL}{c^2} \ll \beta |\Delta T|} \quad \text{if } U \sim L/c \quad \text{and} \quad \boxed{\left(\frac{U}{c}\right)^2 \ll \beta |\Delta T|}$$

The continuity eq. states $\frac{1}{\rho} \frac{D\rho}{Dt} = -\text{div} \vec{v}$;

$$\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{1}{\rho} \frac{D\rho}{Ds} \frac{Ds}{Dt} \sim \frac{\beta T}{\rho} \frac{|\Delta T| U}{L} \sim \frac{\beta T}{\rho} \frac{c_p |\Delta T|}{T} \frac{U}{L} = \frac{\beta |\Delta T| U}{L}$$

$$\Rightarrow \frac{1}{\rho} \frac{D\rho}{Dt} \sim \frac{\beta |\Delta T| U}{L} \quad (a)$$

but also $\text{div} \vec{v} \sim U/L$ (b)

so if (a) \ll (b) holds $\Rightarrow \frac{\beta |\Delta T| U}{L} \ll U/L$, then the two terms are of different order of magnitude and must vanish separately, which corresponds to the condition $\text{div} \vec{v} = 0$.

Finally, let us go back to the inequality expressing negligible mechanical compressibility

$$|\Delta p| \ll \frac{\rho c_p |\Delta T|}{\beta T} \quad \text{and with } |\Delta p| \sim \rho U L / \tau \Rightarrow$$

$$\frac{U L}{\tau} \ll \frac{c_p |\Delta T|}{\beta T} \quad \text{and if } U \sim L/\tau \text{ or } \tau \sim L/U \Rightarrow \frac{U^2}{\tau} \ll \frac{c_p |\Delta T|}{\beta T}$$

We have therefore all conditions expressing incompressibility, $U \ll c$, $\tau \ll L/U$, $\beta |\Delta T| \ll 1$, $U^2 \ll \frac{c_p |\Delta T|}{\beta T}$.

Once we can then justify the approximation $\frac{Ds}{Dt} = \frac{c_p}{T} \frac{DT}{Dt}$, we can use the heat eq., so

$$\frac{DT}{Dt} = \frac{T}{\rho c_p} \frac{Ds}{Dt} = \frac{T}{\rho} \left(\frac{1}{\rho T} \sigma_{ij}' \partial_j v_i - \frac{1}{\rho T} \text{div} \vec{q} \right) = \frac{1}{\rho c_p} \sigma_{ij}' \partial_j v_i - \frac{1}{\rho c_p} \text{div} (-K \text{grad} T)$$

and defining $\chi \equiv K/\rho c_p$ THERMAL DIFFUSIVITY [χ] = (m²/s) (also called α),

with $\sigma_{ij}' \partial_j v_i = \frac{\eta}{2} (\partial_j v_i + \partial_i v_j)^2$ for Newtonian fluids and incompressible flows,

we get

$$\boxed{\frac{DT}{Dt} = \frac{\eta}{2 \rho c_p} (\partial_j v_i + \partial_i v_j)^2 + \chi \nabla^2 T} \quad \text{heat equation (associated with } \text{div} \vec{v} = 0 + \text{b.c.)}$$

For a fluid at rest or a solid - practically speaking, in any purely conductive situation, where no advection and thus no convection (transfer of heat with the continuum motion) takes place -

$D/Dt \rightarrow \partial/\partial t$ and $(\partial_j v_i + \partial_i v_j) = 0$ and the eq. reduces to

$$\boxed{\frac{\partial T}{\partial t} = \chi \nabla^2 T} \quad \text{heat equation for pure conduction (with its b.c.)}$$

This eq. is good for solids, not so much for fluids (where heat is transferred mostly by convection). In a steady-state situation, it reduces to a Laplace eq. $\nabla^2 T = 0$.