Boundary conditions for the heat equation

Boundary conditions tell us the nature of the interaction between our system and the outside world. We shall discuss the b.c. for a specific problem, i.e. a flat (planar) solid layer, unbounded in two directions (x and y), topped by a surface that is an interface to the outside. We have in mind the Earth's crust with this example, specifically, and imagine the atmosphere on top. The problem shall hence be one-dimensional with the interface at $z=\phi$ (and if the crust is a layer of finite depth there shall be a bottom interface, too, but here we focus first on the interaction with the atmosphere). A range of b.c. is possible, from cruder to more accurate.

⊙ Dirichlet b.c.: That is setting $T(z=\phi)=T_{sup}$ to a specific (possibly time-dependent) value. In general it may be ok, within our geophysical example it it quite a rough approximation since T_{sup} is not defined a priori (e.g., imparted by an infinite heat reservoir) but it is unknown and part of the problem itself.

⊙ Neumann b.c.: This is a condition on the derivative of T normal to the surface, in mathematical terms. Physically, this means setting a condition on the heat flux:

$$q_z = -K \frac{\partial T}{\partial z}\bigg|_{z=\phi} = \text{assigned.}$$

It is more meaningful than the Dirichlet b.c., but still incomplete, since it requires an exact knowledge of the value of q_z.

⊙ Robin b.c.: Mathematically, it is in a form where both the unknown and its normal derivative appear. In our case, we state an equality on the ground ($z=\phi$) between

conductive heat flux coming from the crust = convective heat flux of the atmosphere

$$-K\frac{\partial T}{\partial z}\bigg|_{z=\phi} = h\left[T(\phi,t) - T_0 f(t)\right]$$

where $T(\phi,t)$ is the temperature of the crust at the interface, and $T_0 f(t)$ is the temperature of the atmosphere at the interface, where $f(t)$ accounts for a, say, daily or seasonal periodic variation of the atmosphere conditions. The right-hand side is in the form of a convective flux, with h thermal convection coefficient, since convection is much more significant than conduction for heat exchange in a gas.

⊙ The conductive heat flux q_z may be set as equal to a more complex expression on the right-hand side of the flux balance equation, where indeed other physical phenomena should be taken into consideration.

* Absorption of the heat flux coming from the Sun

This is actually a dominant term; one can approximate it to an oscillating (seasonal) heat flux whose magnitude depends on many factors, e.g. the latitude. It is always an absorbed quantity (hence a minus sign): $Q_\odot(t) = -Q_{\odot o} f(t) = -Q_{\odot o} \exp(-i\omega t + \varphi)$
where $f(t)$ is more or less complicated as a function of time depending on the period and resolution we want to take into account (daily variations are definitly not sinusoidal).

* Radiative heat emission

Another very significant term, it takes into account the release of heat stored by the Earth, which can be said to emit in the infrared with an average temperature of 15°C. With a quite crude approximation as a black body we have a (positive, as emitted from the crust) term

$$q_r(t) = \sigma T^4(\varphi, t) \quad \text{with } \sigma \text{ Stefan-Boltzmann constant}$$
$$(\sigma = 5.67 \cdot 10^{-8}\ W/m^2K^4)$$

* Absorption of radiative heat

A part of the emitted radiation can be retained in the atmosphere and given back, so that there is a term similar to the previous one, but with opposite sign:

$$q_{abs}(\cdot) = -\tilde{\sigma} T_{env}^4$$

where T_{env} is the environment temperature, $\tilde{\sigma}$ a constant depending on the specific properties of the local environment. Since the radiative heat is in the infrared range, the absorption by the atmosphere is essentially due to water vapour and clouds, as atmospheric gases like O_2 and N_2 cannot absorb/release radiation in this range. Notice indeed that the temperature drops much less at night in the presence of humidity, i.e. in cloudy nights with respect to clean, dry nights, the extreme case being dry deserts (where hot days alternate to very cold nights). Actually, besides humidity (H_2O) other gases have a dramatic effect on this contribution and are called <u>greenhouse gases</u> due to their warming effect (CO_2, CH_4, N_2O, O_3). Without them the average temperature at the ground would be below 0°C!

When the difference $T(\varphi,t) - T_{env}(t)$ is not large, the sum of radiative emission and absorption can be linearized as

$$G\left[T(\varphi, t) - T_{env}(t)\right]$$

with a constant G depending on the features of the radiation and the environment (definitely

$G \neq h$ simple thermal convection coefficient).

* Latent heat release/absorption

This is a term of significant effect, but only at very limited and specific periods of time during the day: Dusk and dawn (hence the influence extends to a thin crust layer only). Indeed when the sun starts/stops sending heat to a certain region of the Earth, there will be absorption/release of energy by the humidity in the atmosphere associated to a phase transition (at constant T, hence the name of latent heat for this energy).

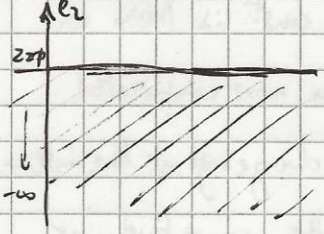
- At dawn, latent heat of vaporization is also subtracted from the ground $\rightarrow$ T drops
- At dusk, latent heat of condensation is released and is available for absorption to the ground.

Dew is observed at dusk and dawn, associated to these phenomena.

In the following we shall discuss some one-dimensional examples with the simplest b.c., like Dirichlet or Robin (conductive-convective) conditions, limited to analytically solvable cases both in a steady state and with a transient term (i.e. an initial condition whose effect attenuates in time leaving the stage to a steady-state solution).

When considering a finite-depth crust, we shall have a b.c. at the bottom, and a reasonable but easy choice is assuming a geothermal heat flux (heat released by the interior of the Earth) as assigned.

Infinitely deep crust with Dirichlet b.c. – Steady-state solution



We consider the system unbounded and invariant in x and y, therefore a one-dimensional problem

$$\frac{\partial T(z,t)}{\partial t} = \chi \frac{\partial^2 T(z,t)}{\partial z^2}$$

with a b.c. on the interface between the solid crust and the upper atmosphere that is a set temperature ($\Rightarrow$ a Dirichlet b.c.) featuring a periodic time oscillation:

$$T(z=0,t) = T_0 \exp(-i\omega t) \, ;$$

hence we look for a solution respecting the invariances of the problem with the separation of variables

$$T(z,t) = \psi(z) e^{-i\omega t}$$ which we plug into the heat eq., getting

$$-i\omega \psi(z) e^{-i\omega t} = \chi \frac{d^2\psi}{dz^2} e^{-i\omega t}$$

$$\Rightarrow \frac{d^2\psi}{dz^2} = -\frac{i\omega}{\chi}\psi(z) = \lambda^2 \psi(z)$$ by defining $\lambda^2 \doteq -i\omega/\chi$; the general solution reads

$$T(z,t) = \left[A e^{\lambda_+ z} + B e^{\lambda_- z}\right] e^{-i\omega t}$$ where $\lambda_\pm = \pm\sqrt{-i\omega/\chi} = \pm(1-i)\sqrt{\omega/2\chi} = \pm(1-i)\delta$

$(\sqrt{-i} = (1-i)/\sqrt{2})$

where we define $\delta \doteq \sqrt{2\chi/\omega}$ <u>DAMPING DEPTH</u>.

Since T must remain finite for $z \to -\infty$, since $\mathrm{Re}(\lambda_-) < 0$ yielding a real exponential term $\sim \exp[\mathrm{Re}(\lambda_-) z] \to \infty$ for $z \to -\infty$, we must ask for $\underline{\underline{B=0}}$.

$$\Rightarrow T(z,t) = A e^{(1-i) z/\delta} e^{-i\omega t} = A e^{z/\delta} e^{-i(z/\delta + \omega t)}$$

and by applying the b.c in $z=0$,

$$T(0,t) = A e^{-i\omega t} = T_0 e^{-i\omega t} \Rightarrow \boxed{A = T_0}$$ $\Rightarrow$ the complete solution reads

$$T(z,t) = T_0 e^{z/\delta} e^{-i(z/\delta + \omega t)}$$

where we can see a <u>damping with increasing depth</u> and a <u>phase shift in the oscillation</u> between the surface and the points at any given depth (there can be points deep in the crust at their maximum when, e.g., the temperature is at its minimum on the surface); the depth-dependent phase shift is $\Phi_D = z/\delta$ in radians (or $\Phi_D/\omega = z/\sqrt{2\omega\chi}$ in terms of time).

Infinitely deep crust with **Robin** b.c. – steady-state solution

Now our x- and y- unbounded and invariant problem with the interface between an infinitely deep crust and the upper atmosphere is at $z=\phi$ is assigned a more complicated (and physically meaningful) b.c. setting a conductive-convective heat exchange at the interface: The conductive heat flux coming from the depths is equal to the convective flux released in the atmosphere. Hence the problem is stated as follows:

$$\begin{cases} \dfrac{\partial T}{\partial t} = \chi \dfrac{\partial^2 T}{\partial z^2} \\ q_z = -K \dfrac{\partial T}{\partial z}\Big|_{z=\phi} = h\left(T(\phi,t) - T_0 e^{-i\omega t}\right) \end{cases}$$

with h convective heat exchange coefficient, $T(\phi,t)$ = temperature of the crust at the surface and $T_0 e^{-i\omega t}$ air temperature at the ground, featuring a periodic oscillation (e.g., a seasonal variation). Once again we guess a solution by separation of variables

$$T(z,t) = \psi(z)\, e^{-i\omega t}$$

whose explicit general form, found by plugging it into the heat eq., is as before

$$\psi(z) = A \exp[(1-i) z/\delta] + B \exp[-(1-i) z/\delta] \quad \text{with} \quad \delta \doteq \sqrt{2\chi/\omega}$$

and again $B=\phi$ to guarantee a finite solution for $z \to -\infty$.

Now using the Robin b.c. at $z=\phi$ for $T(z,t) = A\exp[(1-i)z/\delta]\exp(-i\omega t)$

$$-\frac{K(1-i)}{\delta} A \exp[(1-i)z/\delta]\Big|_{z=\phi} e^{-i\omega t} = h\left[A e^{-i\omega t} - T_0 e^{-i\omega t}\right]$$

$$\Rightarrow -K(1-i)A/\delta = h(A - T_0) \quad \Rightarrow A[h + K(1-i)/\delta] = h T_0$$

$$\Rightarrow A = hT_0 / [h + K/\delta - iK/\delta] = hT_0 \frac{(h+K/\delta) + iK/\delta}{(h+K/\delta)^2 + (K/\delta)^2}$$

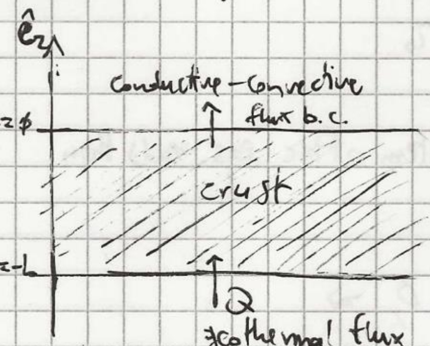
or briefly, in polar form, $A = |A| e^{i\vartheta}$ with $\vartheta = \arctan[\mathrm{Im}(A)/\mathrm{Re}(A)] = \arctan[(K/\delta)/(h+K/\delta)]$ so that we visualize the overall phase shift of the solution

$$T(z,t) = A\exp[(1-i)z/\delta]\exp(-i\omega t) = |A| \underbrace{e^{-z/\delta}}_{\text{damping over depth} \sim \delta} \, e^{-i(\omega t + \underbrace{z/\delta - \vartheta}_{\text{oscillation with phase shift}})}$$

Notice that the phase shift $z/\delta - \vartheta = z\sqrt{\omega/2\chi} - \arctan[(K/\delta)/(h+K/\delta)]$ with respect to the air temperature is present already at the surface $z=\phi$.

Finite-depth crust with a geothermal heat flux - steady-state solution

We now treat the crust as a planar, uniform slab of thickness L; we must then add a b.c. on the bottom surface to the upper one. Let us have a b.c. in the form of a geothermal heat flux from the interior of the Earth (assumed as constant in time). The problem (given the usual x- and y-invariance) is formulated as follows:



$$\begin{cases} \dfrac{\partial T}{\partial t} = \chi \dfrac{\partial^2 T}{\partial z^2} \\ -K \dfrac{\partial T}{\partial z}\Big|_{z=\phi} = h\left[T(\phi,t) - T_0 e^{-i\omega t}\right] \\ -K\dfrac{\partial T}{\partial z}\Big|_{z=-L} = Q \end{cases}$$

The presence of two simultaneous boundary conditions, one of which depends on time while the other one is constant, suggests a steady-state solution in the form of a sum of two functions $\sim f_1(z,t) + f_2(z)$, where $f_1(z,t)$ has a time dependence coming from the time-dependent b.c. and is itself factored in separate-variables terms, while $f_2(z)$ is constant and reflects the present of the constant b.c.; hence we write

$$T(z,t) = \phi(z)e^{-i\omega t} + \psi(z)$$ and inserting it into the heat eq. we get

$$\begin{cases} -i\omega\phi(z)e^{-i\omega t} = \chi \dfrac{d^2\phi}{dz^2}e^{-i\omega t} + \chi\dfrac{d^2\psi}{dz^2} \quad \text{while the b.c. expressions read} \\ -K\dfrac{d\phi}{dz}\Big|_{z=-L} e^{-i\omega t} - K\dfrac{d\psi}{dz}\Big|_{z=-L} = Q \\ -K\dfrac{d\phi}{dz}\Big|_{z=\phi} e^{-i\omega t} - K\dfrac{d\psi}{dz}\Big|_{z=\phi} = h\,\phi(\phi)e^{-i\omega t} - hT_0 e^{-i\omega t} + h\psi(\phi) \end{cases}$$

All equations contain terms both in ϕ and ψ, i.e. oscillatory and constant functions that are linearly independent; therefore both the heat eq. and the b.c. eqs. can be split in two decoupled parts, containing only ϕ and ψ, respectively:

$$① \begin{cases} -i\omega\phi(z)e^{-i\omega t} = \chi\phi''(z)e^{-i\omega t} \\ \phi'(-L) = \phi \\ -K\phi'(\phi)e^{-i\omega t} = h\phi(\phi)e^{-i\omega t} - hT_0e^{-i\omega t} \end{cases} \qquad ② \begin{cases} \psi''(z) = \phi \\ -K\psi'(-L) = Q \\ -K\psi'(\phi) = h\psi(\phi) \end{cases}$$

and we proceed to solve ① and ② separately.

(1) $\phi''(z) = -i\omega/\chi \, \phi(z)$ has a solution in the form already found for the previous examples;

defining $\lambda^2 \doteq -i\omega/\chi$, $\delta \doteq \sqrt{2\chi/\delta}$

$$\underline{\phi(z) = A\exp[(1-i)z/\delta] + B\exp[-(1-i)z/\delta]}$$

while explicitly using this $\phi(z)$ in the b.c. eqs.,

$\phi'(-L) = \phi \;\rightsquigarrow\; (1-i)\exp[-(1-i)L/\delta]A - (1-i)\exp[(1-i)L/\delta]B = \phi$

$-K\phi'(\phi) = h\phi(\phi) - hT_o \rightsquigarrow\; -\frac{K(1-i)}{\delta}A + \frac{K(1-i)}{\delta}B = h(A+B) - hT_o$

let us define the complex number $\alpha \doteq (1-i)L/\delta$; the system of b.c. eqs. reads then

$$\begin{cases} Ae^{-\alpha} - Be^{\alpha} = \phi & \rightsquigarrow \quad B = Ae^{-2\alpha} \\ [h+K\alpha]A + [h-K\alpha]B = hT_o & \quad \text{replacing } B \Rightarrow \end{cases}$$

$$A = hT_o \Big/ \left\{[h+K\alpha] + [h-K\alpha]e^{-2\alpha}\right\}$$

As A is a complex number, summarizing we can say that the solution is some (more or less complicated) oscillating function with a phase shift with respect to the forcing on the upper surface (the periodic air temperature affecting the heat flux balance).

(2) $\psi''(z) = \phi \;\rightsquigarrow\; \psi'(z) = C \;\rightsquigarrow\; \psi(z) = Cz + D$

with b.c. $\begin{cases} -K\psi'(-L) = Q & \rightsquigarrow -KC = Q \\ -K\psi'(\phi) = h\psi(\phi) & \rightsquigarrow -KC = hD \end{cases} \Rightarrow$

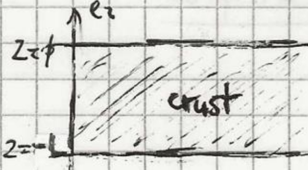
$$\boxed{C = -Q/K}, \quad D = -KC/h \Rightarrow \boxed{D = Q/h}$$

and finally $\underline{\underline{\psi(z) = Q\left(1/h - z/K\right)}}$

i.e. a constant term of the overall solution yielding a negative vertical T gradient given by the heating from the interior of the Earth.

Finite-depth crust - transient solution with homogeneous Dirichlet b.c.

If there is some kind of initial condition to consider, the system is going to experience a transient stage during which the influence of such initial condition attenuates giving place to a steady-state situation determined by non-transient forcings - i.e., the boundary conditions.



We consider again a finite crust layer in $z \in [-L, \phi]$, with x- and y-invariance and homogeneous Dirichlet conditions (= null); more specifically, the temperature profile $T(z,t)$ is expressed as a combination of a steady-state solution $T_{ss}(z,t)$ plus a transient solution $u(z,t)$, where the b.c. have constant zero value - if the constant is not zero, it is sufficient to shift $T_{ss}(z,t)$ by that amount; if different constants occur on different parts of the boundary, the b.c. are inhomogeneous (and we will treat them in the next example). So we write

$$T(z,t) = T_{ss}(z,t) + u(z,t)$$

with conditions on u : ⊙ $u(\phi,t) = u(-L,t) = \phi$ boundary conditions

⊙ $u(z,\phi) = u_0(z)$ initial condition

By separation of variables, $u(z,t) = T(t)\,Z(z)$ and plugging this into the heat eq.

$$\dot{T}(t)\,Z(z) = \chi\, T(t)\,Z''(z)$$

with b.c. $T(t)\,Z(-L) = T(t)\,Z(\phi) = \phi$

and rearranging the heat eq. $\dfrac{\dot{T}(t)}{T(t)} = \chi \dfrac{Z''(z)}{Z(z)}$

which is solved only if both sides are equal to a constant we shall call $\chi\kappa$. Then

⊛ $Z(z)$ must obey the eigenvalue eq.

$$Z''(z) = \kappa\, Z(z) \quad \text{with unknown eigenvalue } \kappa$$

$$+ \; Z(-L) = Z(\phi) = \phi$$

⊛⊛ $T(t)$ must obey $\dot{T}(t) = \kappa\chi\, T(t)$ (an initial-value Cauchy problem)

with κ determined by the eq. for $Z(z)$ and an initial condition we shall consider later.

First of all let us consider ⊛ and determine κ. We must analyze the cases

$\kappa > \phi$; $\kappa = \phi$; $\kappa < \phi$.

+ $K > \phi$ implies a solution $Z(z) = A\exp(\sqrt{K}z) + B\exp(-\sqrt{K}z)$

and the b.c. request $Z(-L) = \phi$ $\Rightarrow$ $\begin{cases} A\exp(-\sqrt{K}L) + B\exp(\sqrt{K}L) = \phi \\ A + B = \phi \end{cases}$

$Z(\phi) = \phi$

with non-trivial solutions if $\det \underline{M} = \phi$ with $\underline{M}$ matrix of $\begin{pmatrix} A \\ B \end{pmatrix}$ coefficients:

$\exp(-\sqrt{K}L) - \exp(\sqrt{K}L) = \phi$ <u>impossible</u> $\Rightarrow$ <u>$K > \phi$ is not an option.</u>

+ $K = \phi$ implies $Z''(z) = \phi \Rightarrow Z(z) = Az + B$

with b.c. $\begin{cases} -AL + B = \phi \\ B = \phi \end{cases}$ $\Rightarrow A = \phi,\ B = \phi$ and trivial null solution

$\Rightarrow$ <u>$K = \phi$ is not an option, either.</u>

+ $K < \phi$ so let us rename it $K = -\lambda^2$ and $Z''(z) = -\lambda^2 Z(z)$ has solution

$$Z(z) = A\sin(\lambda z) + B\cos(\lambda z)$$

and b.c. $\begin{cases} -A\sin(\lambda L) + B\cos(\lambda L) = \phi \\ A\underbrace{\cancel{\sin(\phi)}}_{\phi} + B\underbrace{\cancel{\cos(\phi)}}_{1} = \phi \Rightarrow \boxed{B = \phi} \end{cases}$

and $A\sin(\lambda L) = \phi \rightsquigarrow$ satisfied for discrete values of λ

$\Rightarrow \lambda_n = n\pi/L$ with $n = 1, 2, 3, \dots$

hence $K_n = -\lambda_n^2 = -(n\pi/L)^2$ and the solution is a combination of functions $Z_n(z)$ /

$$Z_n(z) = A_n \sin\left(\frac{\pi n}{L} z\right)$$

We can now reconsider the Cauchy problem for T using K_n: $\dot{T}(t) = K_n \chi T(t)$,

with solutions $T_n(t) = C_n \exp(K_n \chi t) = C_n \exp\left[-\chi (n\pi/L)^2 t\right]$

and renaming A_n the product $A_n C_n$ for brevity, we summarize

$$u_n(z,t) = Z_n(z) T_n(t) = A_n \exp\left[-\chi (n\pi/L)^2 t\right] \sin\left(\frac{n\pi}{L} z\right) \Rightarrow u(z,t) = \sum_{n=1}^{+\infty} u_n(z,t)$$

The solution is actually not complete until we assign the initial condition and determine the coefficients A_n. The initial condition $u(z,\phi) = u_0(z)$ can be expanded in a series of sines in the interval $z \in [-L, \phi]$ (only sines are acceptables as they can vanish at the ends of the period):

$$u_0(z) = \sum_{n=1}^{+\infty} u_{0n} \sin\left(\frac{n\pi}{L} z\right)$$

($n = \phi$ yields a constant, conflicting with the homogeneous b.c. requesting $u = \phi$ at $z = -L, \phi$)

Enforcing the b.c., the solution must be $= u_0(z)$ at $t=0$:

$$u(z,0) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{L} z\right) = u_0(z) = \sum_{n=1}^{+\infty} u_{0n} \sin\left(\frac{n\pi}{L} z\right)$$

$\Rightarrow \quad A_n = u_{0n} \quad \forall n \Rightarrow$

$$\boxed{u(z,t) = \sum_{n=1}^{+\infty} u_{0n} \sin\left(\frac{n\pi}{L} z\right) \exp\left[-\chi\left(\frac{n\pi}{L}\right)^2 t\right]}$$

This solution attenuates and finally vanishes as time goes by: Each term of the series contains a decreasing exponential with its own time constant $\tau_n = \frac{1}{\chi}\left(\frac{L}{n\pi}\right)^2$; the most persistent term (maximum τ_n) is $n=1$ with $\tau_1 = \frac{1}{\chi}(L/\pi)^2$.

As a time of some τ_1 the transient part has disappeared and only the steady-state contribution $T_{ss}(z,t)$ survives: But homogeneous b.c. mean the same value on each part of the boundary and, in turn, T_{ss} will hence be homogeneous as well (null or constant T value $\neq 0$).

Finite-depth crust – transient solution with inhomogeneous Dirichlet b.c.

With respect to the previous problem here the only added complication lies in the boundary conditions taking different values at different positions. That is an easily solved issue, since the heat eq. is linear both in the boundary and initial conditions; therefore we can draw the overall solution as the composition of ① a steady-state solution with the inhomogeneous b.c. and ② a transient solution obeying the initial condition and homogeneous (null) b.c. (that is the example before indeed!).

So let us have as an example a finite-thickness slab and write down the problem:

$$\begin{cases} \frac{\partial T(z,t)}{\partial t} = \chi \frac{\partial^2 T(z,t)}{\partial z^2} \\ \left.\begin{array}{l} T(-L,t) = T_2 \\ T(\phi,t) = T_1 \end{array}\right\} \text{b.c.} \\ T(z,\phi) = T_0(z) \quad \text{initial condition} \end{cases}$$

We perform a shift on the temperature variable and solve the problem for $T' = T - T_1$:

$$\begin{cases} \frac{\partial T'(z,t)}{\partial t} = \chi \frac{\partial^2 T'(z,t)}{\partial z^2} \\ T'(-L,t) = T_2 - T_1 \doteq T_3 \\ T'(\phi,t) = \phi \\ T'(z,\phi) = T_0(z) - T_1 \doteq T_4(z) \end{cases}$$

As we discussed above, we decompose the solution into

$$T'(z,t) = \underbrace{T_T(z,t)}_{\text{transient component}} + \underbrace{T_{ss}(z)}_{\text{steady-state component: time-independent since the b.c. are so}}$$

The steady-state solution must hence satisfy the time-independent heat eq. with the b.c. as given:

$$\frac{\partial^2 T_{ss}}{\partial z^2} = \phi \quad ; \quad T_{ss}(-L) = T_3 \; ; \; T_{ss}(\phi) = \phi$$

$$T_{ss}(z) = Az + B \quad + \quad \begin{cases} -AL + B = T_3 \\ \boxed{B = \phi} \end{cases} \rightarrow \boxed{A = -T_3/L} \Rightarrow T_{ss}(z) = -T_3 z/L = \frac{(T_1 - T_2)}{L} z$$

By virtue of linearity, since T_{ss} obeys the b.c., the T_T obeys homogeneous null b.c., together with the initial conditions:

$$\frac{\partial T_T}{\partial t} = \chi \frac{\partial^2 T_T}{\partial z^2} \quad ; \quad T_T(-L,t) = T_T(\phi,t) = \phi \qquad \text{b.c.}$$

$$T_T(z,\phi) = T_{0T}(\phi) = T_A(z) - T_{SS}(z) \qquad \text{i.c.}$$

We have already found the solution to the transient problem with homogeneous b.c.:

$$T_T(z,t) = \sum_{n=1}^{+\infty} T_{0T_n} \sin\left(\frac{n\pi}{L} z\right) \exp\left[-\chi\left(\frac{n\pi}{L}\right)^2 t\right]$$

where the coefficients T_{0T_n} are obtained through the series expansion of the i.c.

$$T_{0T}(z) = T_A(z) - T_{SS}(z) = T_T(z,\phi) = \sum_{n=1}^{+\infty} T_{0T_n} \sin\left(\frac{n\pi}{L} z\right)$$

So finally we have the complete solution:

$$T(z,t) = T'(z,t) + T_1 = T_1 + T_{SS}(z) + T_T(z,t) =$$

$$= T_1 + \frac{T_1 - T_2}{L} z + \sum_{n=1}^{+\infty} T_{0T_n} \sin\left(\frac{n\pi}{L} z\right) \exp\left[-\chi\left(\frac{n\pi}{L}\right)^2 t\right]$$

which, over a time long enough (determined by the time constants of the exponential) reduces to the pure steady-state part

$$T(z,t) \rightsquigarrow T_1 + \frac{T_1 - T_2}{L} z$$