# Heat equation: Green's function method (for an unbounded medium)

We have seen relatively easy examples, and more of those exist, where the heat equation can be solved by means of easy techniques (e.g. separation of variables, use of combination of different terms). A more general method involves the use of a Green's function (or propagator). We shall present the simplest application of this method, as a conceptual example – more can be found in numerous books, e.g. Landau & Lifshitz, and others. The example considers an unbounded medium, i.e. we have the whole $\mathbb{R}^3$ as a domain. If the domain is limited, the type of b.c. (Dirichlet, Neumann) will determine the specific variant of the method to be used (where in contrast to a Fourier transform the Kernel or nucleus of the integral transform is not an imaginary exponential, but a sine or cosine only).

The structure of the problem is formally similar to many others (including e.g. the Schrödinger eq.), i.e.

$$\begin{cases} \dfrac{\partial f(\bar{x},t)}{\partial t} = \chi \nabla^2 f(\bar{x},t) & \text{where for us } f(\bar{x},t) = T(\bar{x},t) \\ + \quad f(\bar{x},\phi) = f_0(\bar{x}) & \text{(initial conditions)} \\ + \quad f(\bar{x}\to\infty, t) \text{ must be limited} & \text{(b.c.)} \end{cases}$$

The Fourier transform is defined as $$\hat{f}(\bar{p},t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\bar{p}\cdot\bar{x}} f(\bar{x},t)\, d^3x \qquad \text{(FT)}$$

with its inverse transform as $$f(\bar{x},t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i\bar{p}\cdot\bar{x}} \hat{f}(\bar{x},t)\, d^3p \qquad \text{(IFT)}$$

By application of the FT we get to a more manageable ordinary differential eq.; so by multiplying the heat eq. by $\frac{1}{(2\pi)^{3/2}} e^{-i\bar{p}\cdot\bar{x}}$ and integrating over the whole domain $\mathbb{R}^3$ we get

$$\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\bar{p}\cdot\bar{x}} \frac{\partial f(\bar{x},t)}{\partial t} d^3x = \underline{\underline{\frac{\partial \hat{f}(\bar{p},t)}{\partial t}}} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\bar{p}\cdot\bar{x}} \chi \nabla^2 f(\bar{x},t)\, d^3x = \text{by the properties of the FT}^*$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} -\chi|\bar{p}|^2 e^{-i\bar{p}\cdot\bar{x}} f(\bar{x},t)\, d^3x = -\chi|\bar{p}|^2 \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\bar{p}\cdot\bar{x}} f(\bar{x},t)\, d^3x = \underline{\underline{-\chi|\bar{p}|^2 \hat{f}(\bar{p},t)}}$$

$* = FT\left\{\frac{df(x,t)}{dt}\right\} = ip \cdot FT\{f(x,t)\}$ is quickly proven in 1D; integrating by parts $\int_{-\infty}^{+\infty} e^{-ipx} \frac{df}{dx} dx =$ $= \left[e^{-ipx} f(x,t)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -ip e^{-ipx} f(x,t)\, dx$ and $\left(e^{-ipx} f(x,t)\right)_{-\infty}^{+\infty} = \phi$ since this function is absolutely integrable (the integral of the absolute value over $\mathbb{R}$ is finite). By recursive application, $FT\left\{\frac{d^n f}{dx^n}\right\} = (ip)^n FT\{f(x,t)\}$. Easily extended to 3D.

So now our problem becomes a Cauchy initial value problem

$$\begin{cases} \dfrac{\partial \hat{f}(\bar{p},t)}{\partial t} = -\chi p^2 \hat{f}(\bar{p},t) \\ \hat{f}(\bar{p},0) = \hat{f}_0(\bar{p}) \quad \text{TF of the initial condition} \end{cases}$$

with known solution $\hat{f}(\bar{p},t) = \hat{f}_0(p)\, e^{-\chi p^2 t}$ with inverse transform

$$f(\bar{x},t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i\bar{p}\cdot\bar{x}} e^{-\chi p^2 t} \hat{f}_0(\bar{p})\, d^3p = \left[\text{inverting } \hat{f}_0(\bar{p}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\bar{p}\cdot\bar{x}'} f_0(\bar{x}')\, d^3x'\right]$$

$$= \int_{\mathbb{R}^3} \underbrace{\frac{1}{(2\pi)^3} \left[\int_{\mathbb{R}^3} e^{i\bar{p}\cdot(\bar{x}-\bar{x}')} e^{-\chi p^2 t}\, d^3p\right]}_{G(\bar{x}-\bar{x}',t)} f_0(\bar{x}')\, d^3x'$$

$$\Rightarrow \quad f(\bar{x},t) = \int_{\mathbb{R}^3} G(\bar{x}-\bar{x}',t) f_0(\bar{x}')\, d^3x' \quad \text{convolution of } G \text{ and } f_0$$

where we define the Green's function (or kernel, or propagator; also, heat kernel, when specifically talking about the heat eq.)

$$G(\bar{z},t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{(i\bar{p}\cdot\bar{z} - \chi p^2 t)}\, d^3p$$

We can manipulate the function's integral as follows:

$$G(\bar{z},t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3p\, e^{-i\bar{p}\cdot\bar{z}} e^{-\chi p^2 t} = \frac{1}{(2\pi)^3} \prod_{k=1}^{3} \int_{\mathbb{R}} e^{ip_k z_k} e^{-\chi p_k^2 t}\, dp_k =$$

$$\left(\text{with } ip_k z_k - \chi p_k^2 t = -\chi p_k^2 t + ip_k z_k + z_k^2/4\chi t - z_k^2/4\chi t = -\left[p_k\sqrt{\chi t} - iz_k/2\sqrt{\chi t}\right]^2 - z_k^2/4\chi t\right)$$

$$= \frac{1}{(2\pi)^3} \prod_{k=1}^{3} e^{-z_k^2/4\chi t} \int_{\mathbb{R}} e^{-\left[p_k\sqrt{\chi t} - iz_k/2\sqrt{\chi t}\right]^2}\, dp_k = *$$

$$= \frac{1}{(2\pi)^3} \prod_{k=1}^{3} e^{-z_k^2/4\chi t} \sqrt{\pi/\chi t} = \frac{1}{(2\pi)^3} \left(\frac{\pi}{\chi t}\right)^{3/2} e^{-|\bar{z}|^2/4\chi t}$$

$$\Rightarrow \boxed{G(\bar{x}-\bar{x}',t) = \frac{1}{8(\chi t)^{3/2}} e^{-|\bar{x}-\bar{x}'|^2/4\chi t}}$$

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$* =$ Gaussian integral: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$; $\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$ and also $\int_{-\infty}^{+\infty} e^{-(\sqrt{\alpha}x - \beta)^2} dx = \sqrt{\pi/\alpha}$

Some notable remarks:

⊙ The result of the Gaussian integral is $\sqrt{\pi/\chi t}$; the existence of the integral is guaranteed only if $\sqrt{\chi t}$ is real, $\Rightarrow$ $\underline{t \geq \phi}$. For $t < \phi$ the solution has no physical sense. The fact that a solution is possible for time <u>after</u> the initial condition is a sign of the irreversibility of the heat problem. That is not necessarily true for all equations with this formal structure, e.g. the Schrödinger eq. $i\hbar \frac{\partial \psi(\bar{x},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\bar{x},t)$

where if we repeat the process of building the Green's function we find out there is no prescription about $t > \phi$: The solution exists $\forall t \in (-\infty, +\infty)$.

⊙ If the initial condition is a localized source $f_0(\bar{x}) = \delta(\bar{x})$

$$\Rightarrow f(\bar{x},t) = \int_{\mathbb{R}^3} G(\bar{x}-\bar{x}', t)\, \delta(\bar{x}')\, d^3x' = G(\bar{x},t)$$

i.e. the Green's function is the solution of the equation having a delta-localized initial condition; In this sense, it really constitutes the fundamental solution, the building block with which we can construct any other one.

⊙ We should also explicitly prove that the solution

$$f(\bar{x},t) = \int_{\mathbb{R}^3} G(\bar{x}-\bar{x}', t)\, f_0(\bar{x}')\, d^3x'$$

satisfies the initial condition $f_0(\bar{x})$ in the limit $\lim_{t \to \phi} f(\bar{x},t)$, which is not completely trivial, but indeed it can be proven that

$$\lim_{t \to \phi} \frac{1}{2\sqrt{\pi t}} \exp(-x^2/4t) = \delta(x)$$

which ultimately is key in getting to prove the hypothesis $\left( f(\bar{x}, t \to \phi) = \int \delta(\bar{x}-\bar{x}')\, f_0(\bar{x}')\, d^3x' = f_0(\bar{x}) \right)$.

Similarity law for the heat equation

Let us reprise the heat equation in order to get a dimensionless version of it:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\bar{v}\cdot\text{grad})\,T = \chi\nabla^2 T + \frac{\nu}{2c_p}(\partial_j v_i + \partial_i v_j)^2$$

or by calling $S_{ij} \doteq (\partial_j v_i + \partial_i v_j)$, $\Rightarrow \boxed{\frac{DT}{Dt} = \chi\nabla^2 T + \frac{\nu}{2c_p} S_{ij}S_{ij}}$

To make this eq. dimensionless we rescale all variables using characteristic quantities for the specific problem: $L, U, T_1 - T_0$ (we must estimate a characteristic $\Delta T$, as the trigger to heat exchange is not some absolute temperature, but temperature differences):

$\bar{x}' = \bar{x}/L \quad \rightsquigarrow \bar{x} = L\bar{x}' \quad \rightarrow \quad dx = L\,dx'$

$\bar{v}' = \bar{v}/U \quad \rightsquigarrow \bar{v} = \bar{v}'U \quad \rightarrow \quad dv = U\,dv'$

$t' = tU/L \quad \rightsquigarrow t = t'L/U \quad \rightsquigarrow \quad dt = \frac{L}{U}dt'$

$\vartheta = (T - T_0)/(T_1 - T_0) \rightsquigarrow T - T_0 = \vartheta(T_1 - T_0) \rightsquigarrow dT = (T_1 - T_0)\,d\vartheta$

hence the heat eq. is rewritten replacing dimensionless variables

$$\frac{U}{L}\frac{\partial\vartheta}{\partial t'}(T_1 - T_0) + \frac{U}{L}(\bar{v}'\cdot\text{grad}')\,\vartheta\,(T_1 - T_0) = \chi\frac{1}{L^2}\nabla'^2\vartheta\,(T_1 - T_0) + \frac{\nu}{2c_p}\frac{U^2}{L^2}S'_{ij}S'_{ij}$$

and dividing everything by $\frac{U}{L}(T_1 - T_0)$, we get

$$\frac{\partial\vartheta}{\partial t'} + (\bar{v}'\cdot\text{grad}')\,\vartheta = \underbrace{\chi\frac{1}{UL}}_{*}\nabla'^2\vartheta + \underbrace{\frac{\nu}{2c_p}\frac{U}{L}\frac{1}{(T_1 - T_0)}}_{**}S'_{ij}S'_{ij}$$

$$* \quad \frac{\chi}{UL} = \frac{\chi}{\nu}\frac{\nu}{UL} = \frac{\chi}{\nu}\frac{1}{Re} = \frac{1}{Pr}\frac{1}{Re}$$ having defined $\boxed{Pr \doteq \frac{\nu}{\chi}}$ Prandtl (dimensionless) number

$$** \quad \frac{\nu}{2c_p}\frac{U}{L}\frac{1}{T_1 - T_0} \underset{\uparrow}{=} \frac{\chi}{UL}\left(\frac{1}{\chi}\frac{\nu}{2c_p}\frac{U^2}{T_1 - T_0}\right) \underset{\uparrow}{=} \frac{1}{Re\,Pr}\frac{\eta U^2}{2K(T_1 - T_0)} = \frac{1}{Re\,Pr}\frac{1}{2}Br$$

factoring $1/Pr\,Re$ $\qquad$ using $\chi = K/\rho c_p$

having defined $\boxed{Br \doteq \eta U^2/K(T_1 - T_0)}$ Brinkman number

Finally we get $\boxed{\frac{D\vartheta}{Dt'} = \frac{1}{Re\,Pr}\left(\nabla'^2\vartheta + \frac{1}{2}Br\,S_{ij}S_{ij}\right)}$ dimensionless heat eq.

where the physics is contained inside the three dimensionless numbers Re, Pr, Br.

Let us make a few remarks about the results.

⊙ The Prandtl number $Pr = \nu/\chi$ is a property of the medium – it does not depend on its dynamic state, indeed there is nothing like velocity or density in it. There can be a dependence on the contrary, on the thermodynamic state (temperature). Typical values are in the unitary range for gases, while they can vary enormously for liquids: Common fluids like water and alcohol feature $Pr \sim 1-10$, motor oils span $10^2 - 10^4$, an extreme example is the Earth's mantle with $Pr \sim 10^{21} \div 10^{25}$.

Clearly Pr compares viscous and thermal properties of the medium, and specifically appears to be a ratio between kinematic (viscous) and thermal diffusivity.

The product RePr is also given its own name sometimes: $Pe = RePr = UL/\chi$ Péclet number, a ratio between advective and thermal transport.

⊙ The interpretation of the Brinkman number is made clearer by means of a small manipulation, i.e. multiplying numerator and denominator times $L^2$:

$$\frac{1}{2}Br = \frac{\eta U^2}{2K(T_1-T_0)} = \frac{1}{2}\eta\left(\frac{U}{L}\right)^2 \Big/ \frac{1}{L} K \frac{T_1-T_0}{L}$$

Now let us recall the fact that energy dissipation for an incompressible flow is

$$\frac{dE_m}{dt} = -\frac{1}{2}\eta\int(\partial_j v_i + \partial_i v_j)^2 d^3x \sim \frac{1}{2}\eta\left(\frac{U}{L}\right)^2 \cdot Vol$$

$\Rightarrow \frac{1}{2}\eta(U/L)^2$ is an energy per unit time and volume received by a continuum element through a viscous dissipation process of mechanical energy.

On the other hand, the heat flux is $\bar{q} = -K \,\mathrm{grad}\, T \sim K\Delta T/L$ and $\int \bar{q}\cdot d\bar{\sigma} \sim qL^2$ is a power (energy per unit time), therefore

$qL^2/L^3 = q/L \sim \frac{1}{L} K\frac{\Delta T}{L}$ is an energy per unit time and volume received by a continuum element through heat conduction.

So we can see that Br is a ratio between energies supplied to the continuum by viscous dissipation and heat transfer.

⊙ A critical issue lies in the rescaling process of lengths. Indeed we chose L as a characteristic dynamics length scale, i.e. a typical length of velocity variation. Therefore in the dimensionless expression of the heat eq. we can say that kinematic quantities and their derivatives are really of unitary magnitude, but we cannot guarantee this for T and its derivatives, since the length scale of temperature variation does not need to be the same. As a consequence it is not entirely and always true that the comparison between conduction and viscous terms is left purely to the dimensionless numbers Re, Pr, Br. In the following considerations, we will assume that thermal and dynamic scales are indeed similar, so that $S'_{ij}S'_{ij}$ and $\nabla'^2\vartheta$ are $\sim 1$.

⊙ When $Br \ll 1$ the viscous dissipation is negligible with respect to heat transfer, the heat eq. simplifies to $\frac{D\vartheta}{Dt'} = \frac{1}{RePr}\nabla'^2\vartheta$, the dimensionless version of $\frac{DT}{Dt} = \chi\nabla^2 T$ we have seen to hold for solid media in particular.

⊙ The solution of the heat eq. yields a $\vartheta$ (or $T$) that depends explicitly on RePr, but there also is an implicit dependence on Re, that is on the velocity and dynamic properties in general, through the solution of the Navier-Stokes eq.; hence

$\vartheta = \vartheta(\vec{x}' = \vec{x}/L, t', Re, Pr)$ and in general $T = (T_1 - T_0)\vartheta(\vec{x}', t', Re, Pr)$

(add Br if viscous dissipation cannot be neglected).

⊙ Let us evaluate the limit cases, for large and small Re.

**$Re \to \infty$**

The heat eq., once we assign Pr (a feature of the medium) and Br sees the right-hand side vanishing, so that

$\frac{D\vartheta}{Dt'} = 0$ and since $ds = c_p dT/T \Rightarrow \frac{Ds}{Dt} = 0$ adiabatic eq.

Notice that Re very large implies small $\eta$ ($\nu$); since $Pr = \nu/\chi$ has been set by the medium, small $\nu \to$ small $\chi$ i.e. negligible heat exchange, and that is exactly what is required to have an ideal fluid (heat exchange leads to non-ideality and irreversibility). In other words, we complete here the reasons for the claim, first stated when dealing with the Navier-Stokes eq, that the ideal fluid is the limit of a real fluid for $Re \to \infty$. Indeed in this limit

Navier-Stokes eq. $\longrightarrow$ reduces to Euler's eq.

heat eq. $\longrightarrow$ reduces to adiabatic eq.

therefore the thermodynamic evolution of the fluid, for $Re \to \infty$, becomes reversible. Note that this can also hold at a local level, when we can define a local Reynolds number satisfying this condition in region sufficiently far, e.g., from boundaries and viscous boundary layers.

## $Re \to \emptyset$

In the opposite limit the heat eq. becomes

$$\nabla'^2 \theta + \frac{1}{2} Br\, S'_{ij} S'_{ij} = \emptyset \qquad \text{i.e.} \qquad \chi \nabla^2 T + \frac{\nu}{2c_p} S_{ij} S_{ij} = \emptyset$$

Once again for dominant heat transport we would have

$$\chi \nabla^2 T = \emptyset \rightsquigarrow \nabla'^2 \theta = \emptyset, \quad \text{i.e. } \theta = \theta(\bar{x}' = \bar{x}/L, t') \text{ independent of } Re, Pr!$$

Example: Let us consider a Poiseuille flow in a cylindrical pipe of radius $R$, with the flow induced by $\partial_z p = -\Delta p / l$; a case we already know, where we now also add the conditions $Re \to \emptyset$, $T(R) = T_0$ uniform pipe temperature.

We know the solution for the velocity field (only velocity along the pipe axis $\hat{e}_z$ is present):

$$v_z(r) = \frac{\Delta p}{4\eta l}(R^2 - r^2) = 2\bar{v}\left[1 - (r/R)^2\right]$$

where $\bar{v}$ is the velocity averaged over the transverse cross section $\bar{v} = \frac{1}{\pi R^2}\int_\emptyset^R v_z(r) 2\pi r\, dr = \frac{\Delta p R^2}{8\eta l}$

The heat eq. $\chi \nabla^2 T + \frac{\nu}{2c_p} S_{ij} S_{ij}$ simplifies to

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{dT}{dr}\right) = -\frac{\nu}{2c_p \chi}\left(\frac{\partial v_z}{\partial r}\right)^2$$

and using $\frac{\partial v_z}{\partial r} = -4\bar{v}\frac{r}{R^2} \rightsquigarrow \left(\frac{\partial v_z}{\partial r}\right)^2 = 16\bar{v}^2 \frac{r^2}{R^4}$ so

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{8\nu \bar{v}^2}{c_p \chi R^4} r^3 \rightsquigarrow \frac{dT}{dr} = -\frac{2\nu\bar{v}^2}{c_p \chi R^4} r^3 + \frac{A}{r} \rightsquigarrow T(r) = -\frac{\nu \bar{v}^2}{2c_p \chi R^4} r^4 + A \log r + B$$

$T$ must be finite $\forall r \in [\emptyset, R]$, so $A = \emptyset$; by the boundary condition

$$T(R) = T_0 = -\frac{\nu\bar{v}^2}{2c_p\chi} + B \implies B = \frac{\nu\bar{v}^2}{2c_p\chi} + T_0$$

and finally

$$T(r) - T_0 = \frac{\nu\bar{v}^2}{2c_p\chi}\left[1 - (r/R)^4\right] = \frac{1}{2}\frac{\bar{v}^2}{c_p} Pr\left[1 - (r/R)^4\right]$$

Notice that here $T - T_0 \sim \frac{U^2}{2c_p} Pr \cdot f(\text{geometry})$ (see discussion in the following).

Heat exchange between fluid and solid bodies: Heat transfer coefficient and Nusselt number

We have already seen in previous examples how the heat exchange at the fluid-solid interface can be given in the form of a Robin condition for the heat flux density

$q = \alpha (T_1 - T_0)$ (Newton's law of cooling)

with $\alpha$ = heat transfer coefficient (previously called $h$),

$T_1 = T_s|_{surf}$ = solid temperature at the surface,

$T_0$ = characteristic temperature of the fluid at the interface.

The heat flux at the surface can be written as (matching fluxes in absence of thermally-driven convective mixing)

$q = -K_s \dfrac{\partial T_s}{\partial x_n} = -K_F \dfrac{\partial T_F}{\partial x_n}$ (*) with $T_s$ solid temp., $T_F$ fluid temp., $x_n$ along $\hat{n}$ normal unit vector from solid to fluid,

$K_S$, $K_F$ conduction coefficients for solid and fluid.

Indeed by writing this we have $-K_s \dfrac{\partial T_s}{\partial x_n} = \alpha (T_1 - T_0)$ (**) i.e. the Robin condition we wrote.

The coefficient $\alpha$ can be calculated in principle if we know the temperature distribution and use (*), (**):

$\alpha = \dfrac{-K_F \partial T_F / \partial x_n}{T_1 - T_0}$ = in dimensionless variables $-\dfrac{K_F}{L} \dfrac{\partial \vartheta / \partial x'_n}{\vartheta}$

with $\vartheta = \vartheta(\bar{x}' = \bar{x}/L, Re, Pr)$ and thus $\underline{\alpha = \alpha(\bar{x}' = \bar{x}/L, Re, Pr)}$, i.e. the heat transfer coefficient depends on the properties of the material, the flow and the geometry – plus, on a timescale when we are not dealing with a steady-state problem. Notice that even in a steady-state situation, there is a temperature difference between solid and fluid at the surface, since viscosity causes heat dissipation in the vicinity of the solid boundary. Therefore measuring the temperature in a flow by the insertion of a probe may yield an incorrect reading as friction increases the surface temperature of the probe (perturbative measurement).

The heat transfer coefficient $\alpha$ is not dimensionless, but we can associate a dimensionless number to it:

$\boxed{Nu = \alpha L / K_F}$ Nusselt number ⊛; again, $Nu = Nu(\bar{x}', Re, Pr)$.

For small $Re$ ($Re \to \emptyset$), where the heat eq is simplified to

$\nabla^2 T = \emptyset \Rightarrow T$ is no longer $f(Re, Pr)$ but only a function of geometry, it

[⊛ = Similarly, one can define the Biot number $Bi = \alpha L / K_s$.]

follows that $Nu = Nu(\text{geometry})$ function of geometry only, too; in other words, $Nu$ and thus $\alpha$ can be exactly calculated as for a fluid at rest (well, indeed $Re \to 0$ means very small velocity). Coming back to the issue of temperature measurement, a fluid at rest (or almost) allows the experimenter to perform a correct measure (no flow $\Rightarrow$ no friction).

The problem of a body heated by a fluid flow leads us to further observations and specifically to a comment on the rescaling procedure for the heat equation. As we noted, even in a steady-state flow a temperature difference $T_1 - T_0$ between solid and fluid exists due to the friction in the boundary layer. In the heat eq.

$$\frac{DT}{Dt} = \chi^2 \nabla^2 T + \frac{\nu}{2c_p} S_{ij} S_{ij}$$

the viscous term depending on the velocity gradients $S_{ij}$ can no longer be neglected, and we also note that $T_1 - T_0$ is unknown, so that we must find another way to rescale the temperature. We reason as follows: When the velocity drops from $U$ to $0$ (from a characteristic velocity $U$ in the bulk to zero at the surface, across the distance of the boundary layer), there is a conversion of kinetic energy per unit mass to internal energy by viscous effects,

$$-\Delta E_K \sim +\frac{1}{2}U^2 = +\Delta \mathcal{E} = c_p \Delta T = c_p (T_1 - T_0)$$

therefore we can invert it to obtain a dimensionless temperature $\vartheta$

$$\underline{T - T_0 = (T_1 - T_0)\vartheta = \frac{1}{2}\frac{U^2}{c_p}\vartheta} \;;$$

$$\Rightarrow \frac{DT}{Dt} = \chi \nabla^2 T + \frac{\nu}{2c_p} S_{ij} S_{ij} \quad \text{is rescaled as follows}$$

$$\frac{U}{L}\frac{1}{2}\frac{U^2}{c_p}\frac{D\vartheta}{Dt'} = \frac{\chi}{L^2}\frac{U^2}{2c_p}\nabla'^2\vartheta + \frac{\nu}{2c_p}\frac{U^2}{L^2} S'_{ij} S'_{ij} \quad \text{and by multiplication times } 2c_p L/U^3,$$

$$\frac{D\vartheta}{Dt'} = \underbrace{\frac{\chi}{LU}}_{\frac{1}{Re Pr}} \nabla'^2 \vartheta' + \underbrace{\frac{\nu}{LU}}_{\frac{1}{Re}} S'_{ij} S'_{ij} \Rightarrow \boxed{\frac{D\vartheta}{Dt'} = \frac{1}{Re}\left(\frac{1}{Pr}\nabla'^2\vartheta + S'_{ij}S'_{ij}\right)}$$

As usual, let us evaluate different cases occurring for a range of values of **Re**.

- Intermediate **Re** $\Rightarrow$ the equation cannot be simplified, the right-hand side is $f(Re, Pr) \Rightarrow$ back to real quantities $\underline{T - T_0 = \frac{1}{2}\frac{U^2}{c_p}\vartheta \sim \frac{1}{2}\frac{U^2}{c_p} f(Re, Pr)}$.

⊙ $Re \ll 1$ $\Rightarrow$ the left-hand side can be neglected:

$\frac{D\vartheta}{Dt'} = 0 \Rightarrow \quad \nabla'^2 \vartheta = - Pr\, S'_{ij} S'_{ij}$ a Poisson eq.

The $S'_{ij}$ terms can be determined by solving the Navier-Stokes eq., so they are a function of $Re$; therefore $\nabla'^2 \vartheta \sim Pr \cdot f(Re)$

and consequently $T - T_0 \approx \frac{1}{2}\frac{U^2}{c_p} Pr \cdot f(Re)$ (see previous example Poiseuille flow)

⊙ $Re \gg 1$

Here we can say that the variations in $\bar{v}$, $T$ occur essentially across a thin boundary layer – not necessarily the same: let us call $\delta$, $\delta'$ the velocity and thermal boundary layer, respectively, and show that the features of this limit case essentially depend on $Pr$, i.e. the intrinsic viscous and thermal properties of the fluid.

The power dissipated by viscous friction and released as heat into the fluid is $\sim \eta (U/\delta)^2$ per unit volume; in terms of power per unit area (solid surface), it becomes $\sim \eta U^2/\delta$.

This energy is equal to the heat flux

$$q = -K \frac{\partial T}{\partial x_n} \sim \chi c_p \rho \frac{T_1 - T_0}{\delta'} \quad \text{(also a power per unit area)}$$

$$\Rightarrow \quad \eta U^2/\delta = \chi c_p \rho (T_1 - T_0)/\delta'$$

and thus $T_1 - T_0 = \frac{\nu}{\chi c_p} \frac{\delta'}{\delta} U^2 = \frac{U^2}{c_p} Pr \frac{\delta'}{\delta}$ with $\delta'/\delta$ itself a $f(Pr)$;

we conclude that $T_1 - T_0 = \frac{U^2}{c_p} f(Pr)$