Natural (free) convection

When we studied equilibrium and stability of the atmosphere, i.e. of a fluid subjected to a vertical gravity field, we determined how the stable equilibrium condition involves the temperature T and its gradients: T must be a function of the height z only, and specifically its vertical gradient is bounded as follows:

$$-\frac{dT}{dz} \leq \frac{g\beta T}{c_p} \quad \text{(for dry isentropic air)}$$

If the temperature field depends on the other coordinates or if the T drop is too fast for altitude increase, a <u>free convection</u> or <u>natural convection</u> sets in, i.e. mechanical equilibrium is lost and currents are created that try to make T uniform by mixing of the fluid.

We shall derive the equations that describe this process, based on the following assumptions:

* Mechanically incompressible fluid, i.e. pressure variations lead to negligible density variations.
  We shall see that this request is equivalent to setting a limit to the height of the fluid column (in such a way that pressure variations are not too large).
* Thermal expansion is allowed, on the contrary, i.e. we can accept density variations for T changes (well, that is at the foundations of this phenomenon).
* We consider anyway <u>small</u> variations of ρ and T with respect to the reference values ρ_0, T_0, so that

$$T = T_0 + T'; \quad \rho = \rho_0 + \rho'$$

$$\Rightarrow \rho' = \left(\frac{\partial \rho_0}{\partial T}\right)_p T' = -\rho_0 \underbrace{\left[-\frac{1}{\rho_0}\left(\frac{\partial \rho_0}{\partial T}\right)_p\right]}_{} T' = -\rho_0 \beta T' \quad \text{with } \beta \text{ coefficient of thermal expansion}$$

* Pressure variations are similarly written as

$$p = p_0 + p'$$

where contrary to ρ_0, T_0, p_0 is not a constant reference value but the hydrostatic equilibrium pressure at height z for density and temperature ρ_0, T_0 $\Rightarrow$

$$p_0(z) = -\rho_0 g z + \text{constant} = \rho_0 \bar{g} \cdot \bar{x} + \text{constant} \quad (\text{with } \bar{g} = -g\hat{e}_z).$$

Let us see what kind of constraints come out of these assumptions.

(A) First of all if we consider a scale height h we obtain a hydrostatic pressure drop $\Delta p = \rho_0 g h$ and since $\frac{1}{c^2} = \frac{\partial \rho}{\partial p}(p, s)$ (with c speed of sound in the fluid) we can say that density variations under mechanical stress can be expressed as

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp \quad \text{and thus the finite variation across the height } h \text{ is}$$

$$\Delta\rho = \frac{1}{c^2}\Delta p = \frac{1}{c^2}\rho_0 g h$$

but the initial assumption of negligible mechanical compressibility requires $\Delta\rho \ll \rho_0, \rho' \Rightarrow$

$$\frac{\rho_0 g h}{c^2} \ll \rho' = |\rho_0 \beta T'| = \rho_0 \beta \Delta T \quad \text{across the vertical span } h$$

where $\Delta T = T - T_0$ is a characteristic temperature variation over the scale height h, therefore the condition to be met is

$$\boxed{gh/c^2 \ll \beta \Delta T}$$

(B) All of this has an impact on the way we write the Navier-Stokes eq., too. We have

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\text{grad})\bar{v} = -\frac{1}{\rho}\text{grad}\, p + \nu\nabla^2\bar{v} + \bar{g}$$

where we have to plug in $p = p_0 + p'$, $\rho = \rho_0 + \rho'$; in particular, we manipulate the pressure term:

$$\frac{\text{grad}\, p}{\rho} = \frac{\text{grad}\, p_0}{\rho_0 + \rho'} + \frac{\text{grad}\, p'}{\rho_0 + \rho'} \overset{(*)}{=} \frac{\text{grad}\, p_0}{\rho_0} - \frac{\text{grad}\, p_0}{\rho_0^2}\rho' + \frac{\text{grad}\, p'}{\rho_0} - \frac{\text{grad}\, p'}{\rho_0^2}\rho'$$

We can neglect the last term as it is a second-order infinitesimal. Then we use

$\text{grad}\, p_0 = -\rho_0\, \text{grad}(gz) = \rho_0 \bar{g}$ and $\rho' = -\rho_0 \beta T$, so that

$$\frac{\text{grad}\, p}{\rho} = \bar{g} + \frac{\text{grad}\, p'}{\rho_0} + \beta T' \bar{g} \quad \text{and finally the Navier-Stokes eq. is rewritten as}$$

$$\boxed{\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\text{grad})\bar{v} = -\frac{1}{\rho_0}\text{grad}\, p' + \nu\nabla^2\bar{v} - \beta T'\bar{g}}$$

(C) The heat eq. remains as it is, we just notice that the viscous term is very often found to be negligible, so

$$\frac{\partial T}{\partial t} + (\bar{v}\cdot\text{grad})T = \chi\nabla^2 T \left(+ \frac{\nu}{2c_p} S_{ij} S_{ij} \right)$$

Summarizing, the phenomenon of free convection is described by the modified Navier-Stokes eq., the heat eq., the incompressibility eq. and the boundary conditions; the system of equations thus obtained is called BOUSSINESQ APPROXIMATION (Boussinesq flow).

$$(*) = f(\rho) \simeq f(\rho_0) + \left.\frac{df(\rho)}{d\rho}\right|_{\rho}(\rho - \rho_0), \quad \text{with } f(\rho) = \frac{1}{\rho} \simeq \frac{1}{\rho_0} - \left.\frac{1}{\rho^2}\right|_{\rho=\rho_0}(\rho - \rho_0) = \frac{1}{\rho_0} - \frac{\rho - \rho_0}{\rho_0^2} = \frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2}$$

$$\begin{cases} \dfrac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot \text{grad})\bar{v} = -\dfrac{1}{\rho_0}\,\text{grad}\,p' + \nu \nabla^2 \bar{v} - \beta T' \bar{g} \\ \dfrac{\partial T}{\partial t} + (\bar{v}\cdot\text{grad})T = \chi \nabla^2 T + \dfrac{\nu}{2c_p} S_{ij}S_{ij} \\ \text{div}\,\bar{v} = 0 \\ + \text{b.c.} \end{cases}$$

$\to$ BUOYANCY term: A vertical lift force due to heating and thermal expansion

The system is made out of 5 scalar equations for 5 unknown quantities ($\bar{v}$, p'/ρ_0, T), and thus 5 corresponding parameters exist that enclosed the physics of the specific problem at hand: ν, χ, βg and the characteristic scales h and ΔT. Once again we can fruitfully rescale the equations to their dimensionless versions to highlight their most significant properties.

First of all, we should notice that there is no independent velocity scale u, since the motion is not the effect of external mechanical actions but has its origins in the inhomogeneous heating, leading to convective currents. Hence we derive a characteristic velocity through the comparison of advective and buoyancy terms:

$$(\bar{v}\cdot\text{grad})\bar{v} \sim u^2/h \quad ; \quad \beta T' g \sim \beta \Delta T g \quad \Rightarrow \quad \boxed{u \sim (\beta g h \Delta T)^{1/2}}$$

The other quantities are rescaled as usual:

$$\bar{x}' \doteq \bar{x}/h \; ; \quad \pi \doteq p'/\rho_0 u^2 \; ; \quad \vartheta \doteq T'/\Delta T \; ; \quad t' \doteq t/(h/u)$$

and plugged into the Navier-Stokes eq. to yield

$$\frac{u^2}{h}\frac{\partial \bar{v}'}{\partial t'} + \frac{u^2}{h}(\bar{v}'\cdot\text{grad}')\bar{v}' = -\frac{1}{\rho_0}\rho_0 u^2 \frac{1}{h}\,\text{grad}'\pi + \nu \frac{u}{h^2}\nabla'^2 \bar{v}' + \beta g \Delta T \vartheta \hat{e}_z$$

and multiplying both sides by h/u^2 we get (consider that $\beta g \Delta T \frac{h}{u^2} = 1$)

$$\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}'\cdot\text{grad}')\bar{v}' = -\text{grad}\,\pi + \frac{\nu}{hu}\nabla'^2 \bar{v} + \vartheta \hat{e}_z$$

and the only (dimensionless) coefficient left is

$$\frac{\nu}{hu} = \frac{\nu}{\sqrt{\beta g h^3 \Delta T}} \; ; \quad \text{we define} \quad \boxed{Gr \doteq \frac{\beta g \Delta T h^3}{\nu^2}}$$ Grashof number, and get to

$$\boxed{\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}'\cdot\text{grad}')\bar{v}' = -\text{grad}\,\pi + \frac{1}{(Gr)^{1/2}}\nabla'^2 \bar{v}' + \vartheta \hat{e}_z}$$

We operate similarly on the heat equation, including the viscous term for the sake of completeness:

$$\frac{u}{h}\Delta T\frac{\partial\vartheta}{\partial t'}+\frac{u\Delta T}{h}(\vec{v}'\cdot\text{grad}')\vartheta=\chi\frac{\Delta T}{h^2}\nabla'^2\vartheta+\frac{\nu}{2c_p}\frac{u^2}{h^2}S_{ij}'S_{ij}'$$

and multiply both sides of the eq. by $h/u\Delta T$, obtaining

$$\frac{\partial\vartheta}{\partial t'}+(\vec{v}'\cdot\text{grad}')\vartheta=\frac{\chi}{uh}\nabla'^2\vartheta+\frac{\nu}{2c_p}\frac{u}{h\Delta T}S_{ij}'S_{ij}' \quad ;$$

let us manipulate the two dimensionless groups at the right-hand side,

⊛ $\frac{\chi}{uh}=\frac{\chi}{\nu}\frac{\nu}{uh}=\frac{1}{Pr}\frac{\nu}{\sqrt{\beta g h^3\Delta T}}=\frac{1}{Pr}\frac{1}{(Gr)^{1/2}}$

⊛ $\frac{\nu}{2c_p}\frac{u}{h\Delta T}=\frac{\nu}{2c_p}\frac{K}{hu}\frac{u^2}{K\Delta T}=\frac{1}{2}\underbrace{\frac{K}{\rho c_p}}_{=\chi}\frac{1}{hu}\frac{\eta u^2}{K\Delta T}=\frac{1}{2}\frac{\chi}{hu}Br=\frac{1}{Pr(Gr)^{1/2}}\cdot\frac{1}{2}Br$

where the Brinkman number is defined here using the "thermal" velocity,

$$Br=\frac{\eta u^2}{K\Delta T}=\frac{\eta\beta g h\Delta T}{K\Delta T}=\eta\beta g h/K$$

so that we finally get

$$\boxed{\frac{\partial\vartheta}{\partial t'}+(\vec{v}'\cdot\text{grad}')\vartheta=\frac{1}{Pr(Gr)^{1/2}}\left(\nabla'^2\vartheta+\frac{1}{2}Br\,S_{ij}'S_{ij}'\right)}$$

let us make some comments about these dimensionless expressions.

⊛ First of all we address the Grashof dimensionless number and its physical meaning. A fluid volume that has received some heat takes on a density ρ and experiences a buoyancy force

$$F_b=(m-m_0)g=(\rho-\rho_0)g\ell^3=\rho' g\ell^3=\rho_0\beta\Delta T g\ell^3$$

so if we rewrite Gr as

$$Gr=\beta g\Delta T\ell^3/\nu^2=F_b/\rho_0\nu^2$$

we must have a force at the denominator, too; hence $\rho_0\nu^2$ is an index of <u>viscous forces</u> and

$$Gr=\frac{\text{buoyancy force (of thermal origin)}}{\text{viscous force}}$$

i.e. Gr is a ratio of the force setting the fluid elements in motion (convection) due to heating, to the viscous forces opposing that motion.

⊛ We established a velocity scale of the convective motion $u\sim(\beta g h\Delta T)^{1/2}$. We can notice that

$$\underline{Gr}=\frac{\beta g\Delta T h^3}{\nu^2}=\frac{u^2h^2}{\nu^2}=\left(\frac{uh}{\nu}\right)^2=\text{formally}\doteq\underline{Re^2}$$

that is to say, Gr plays the role of Re where the motion has a purely thermal origin (whereas Re had a source in mechanical forces inducing the flow). Indeed $(Gr)^{1/2}$ is placed in the Navier-

Stokes and heat equations exactly in the same positions as Re.

⊛ The analogy $Gr \sim Re^2$ goes further: At very large Gr ($>10^7 - 10^9$) the motion becomes turbulent; moreover, in the limit $Gr \to \infty$ the Navier-Stokes and heat eqs. are reduced to the Euler and adiabatic eqs, respectively, i.e. we recover the ideal fluid case.

⊛ We considered a purely free, natural convection, where no other source of motion is present besides heating; when mechanical forces concur to setting the fluid in motion, we can talk of FORCED CONVECTION, and the ratio of thermal and mechanical forces is indeed expressed through Gr and Re, identifying a new dimensionless number Ri

$$Ri = \frac{Gr}{Re^2} = \frac{\text{buoyancy term}}{\text{flow shear term}} \quad \text{Richardson number}$$

$Ri = Gr/Re^2 \gg 1$ free convection

$Ri = Gr/Re^2 \sim 1$ mixed convection

$Ri = Gr/Re^2 \ll 1$ forced convection

⊛ Let us give an example showing that the viscous term $S_{ij}S_{ij}$ in the heat eq. is negligible in a realistic situation; we consider water at ambient temperature. With $\eta = 10^{-3}\,Pa \cdot s$, $K = 0.6\ W/m \cdot K$, $\beta \simeq 2 \cdot 10^{-4}\ K^{-1}$ we get $Br = \eta\beta g/K \cdot h \simeq 10^{-6} \cdot h$, so that for a column of water as high as several meters, the viscous term is $\ll 1$ and negligible with respect to the $\nabla'^2\vartheta$ term.

⊛ In a classic instance of almost esoteric synthesis, Landau remarks that different combinations of dimensionless numbers can be created from the parameters occurring in the eqs. for the Boussinesq flow. A suggestion is, as an alternative to Gr,

$$\underline{\underline{Ra}} = \frac{\beta g \Delta T h^3}{\nu\chi} = \frac{\nu}{\chi}\frac{\beta g \Delta T h^3}{\nu^2} = \underline{Pr \cdot Gr} \quad \text{Rayleigh number}$$

which we now comment upon to lift the veil of Landau's hermeticism. Intuitively there is a comparison between a convective phenomenon due to heating (numerator) and a conductive, or diffusive, phenomenon hinted at by diffusivity χ (denominator).

More specifically, let us consider that the convective motion is fueled by the temperature gradient and hindered by viscosity; we can give an estimate of the convective velocity through a comparison of the terms $\nu\nabla^2\bar{v}$ and $\beta T'\bar{g}$ in the Boussinesq eq.:

$$\nu\nabla^2\bar{v} \sim \nu u/h^2 \quad ; \quad \beta T'\bar{g} \sim \beta g \Delta T$$

$$\hookrightarrow u \sim \beta g \Delta T h^2/\nu$$

hence we get $\underline{\tau_{conv} \sim h/u \sim \nu/\beta g \Delta T h}$ characteristic time scale of the thermal transport through convection. (+)

We can estimate the time scale of thermal transport through conduction by another comparison,

$$\frac{\partial T}{\partial t} \sim T/\tau_{diff} \quad ; \quad \chi\nabla^2 T \sim \chi T/h^2$$

$$\Rightarrow \quad \tau_{diff} \sim h^2/\chi$$

and we find out that the Rayleigh number is a ratio of these two time scales:

$$Ra = \frac{\tau_{diff}}{\tau_{conv}} = \frac{\beta g \Delta T h^3}{\chi\nu},$$

or equivalently a ratio of convective to conductive flux (the higher Ra, the larger the convective heat flux).

⊛ If we write the eqs. for the Boussinesq flow in their dimensionless version, we can say that the solutions are

$$\bar{v}(\bar{x},t) = u\,\bar{v}'\left(\bar{x}/h,\ t/(h/u),\ Gr,\ Pr\right)$$

$$T(\bar{x},t) = T'(\bar{x},t) + T_0 = T_0 + \Delta T\,\vartheta\left(\bar{x}/h,\ t/(h/u),\ Gr,\ Pr\right)$$

(where Ra could also be used instead of Gr). Notice that $\bar{v}$ is a function of Pr, too, even if Pr does not appear explicitly in the Boussinesq eq., but comes into play through ϑ, a $f(Pr)$ explicitly through the heat equation.

Also notice that by u we always mean the derived velocity scale $u = \sqrt{\beta g \Delta T h}$; Landau uses a different rescaling u':

$$\underline{u' = \nu/h};$$

this rescaling is not infrequent and it is perfectly legitimate; nonetheless, if we compare the two different velocity scales we have

$$u'^2/u^2 = \frac{\nu}{h}\Big/\beta g \Delta T h^2 \nu = \frac{\beta g \Delta T h^3}{\nu^2} = Gr$$

(+) = The comparison of these term truly represents a comparison of viscous forces $\bar{F}_v$ to the buoyancy force $\bar{F}_b$;

$F_v \sim \underset{\text{shear stress}}{\tau} h^2 \sim \eta \partial_z v\, h^2 \sim \eta u h$; $F_b \sim \Delta\rho h^3 g \sim \rho\beta\Delta T g h^3$; compare $\eta u h \sim \rho\beta\Delta T g h^3 \Rightarrow u \sim \beta g \Delta T h^2/\nu$

which means the dimensionless velocity $\bar{v}'$ is not guaranteed to be of order 1 — it is $v' \sim \sqrt{Gr}$ — and so are the derivatives of velocity. Indeed if we rescale the Boussinesq eq. with $u' = \nu/h$ we get

$$\frac{u'^2}{h}\frac{\partial \bar{v}'}{\partial t'} + \frac{u'^2}{h}(\bar{v}'\cdot \mathrm{grad}')\bar{v}' = -\frac{1}{\rho_0}\rho_0 \frac{u'^2}{h}\mathrm{grad}'\bar{\pi} + \nu\frac{u'}{h^2}\nabla'^2\bar{v}' + \beta g \Delta T \vartheta \hat{e}_z$$

and multiplying both sides by $\frac{h}{u'^2} = \frac{h^3}{\nu^2}$ we get

$$\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}'\cdot \mathrm{grad}')\bar{v}' = -\mathrm{grad}'\bar{\pi} + \nabla'^2\bar{v}' + \frac{\beta g \Delta T h^3}{\nu^2}\vartheta \hat{e}_z \quad \text{that is to say}$$

$$\boxed{\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}'\cdot \mathrm{grad}')\bar{v}' = -\mathrm{grad}'\bar{\pi} + \nabla'^2\bar{v}' + Gr\,\vartheta \hat{e}_z}$$

where it is apparent that if ϑ is of order 1, the other terms, with $\bar{v}'$ and its derivatives, scale with Gr (i.e. definitely not ≈ 1, in general).

The same rescaling yields, for the heat eq.,

$$\frac{u'}{h}\Delta T\frac{\partial \vartheta}{\partial t'} + \frac{u'}{h}\Delta T(\bar{v}'\cdot \mathrm{grad}')\vartheta = \frac{\chi}{h^2}\Delta T\nabla'^2\vartheta + \frac{\nu}{2c_p}\frac{u'^2}{h^2}S'_{ij}S'_{ij}$$

and multiplying both sides by $h/u'\Delta T$ we get

$$\frac{\partial \vartheta}{\partial t'} + (\bar{v}'\cdot \mathrm{grad}')\vartheta = \frac{\chi}{u'h}\nabla'^2\vartheta + \frac{\nu}{2c_p}\frac{u'}{h\Delta T}S'_{ij}S'_{ij}$$

and, with $u' = \nu/h$, $\frac{\chi}{u'h} = \frac{\chi}{\nu} = \frac{1}{Pr}$, $\frac{\nu}{2c_p}\frac{u'}{h\Delta T} = \frac{1}{2}\nu\rho\overbrace{\left(\frac{1}{\rho c_p}\right)}^{\chi/k}\frac{\rho u'}{h\Delta T} = \frac{1}{2}\frac{\chi}{\nu}\frac{\rho u'^2}{k\Delta T} = \frac{1}{2}\frac{1}{Pr}Br$ $\quad \frac{1}{h} = \frac{u'}{\nu}$

$$\boxed{\frac{\partial \vartheta}{\partial t'} + (\bar{v}'\cdot \mathrm{grad}')\vartheta = \frac{1}{Pr}\left(\nabla'^2\vartheta + \frac{1}{2}Br\,S'_{ij}S'_{ij}\right)}$$

(if we redefine the Brinkman number using the alternative rescaling $u' = \nu/h$).