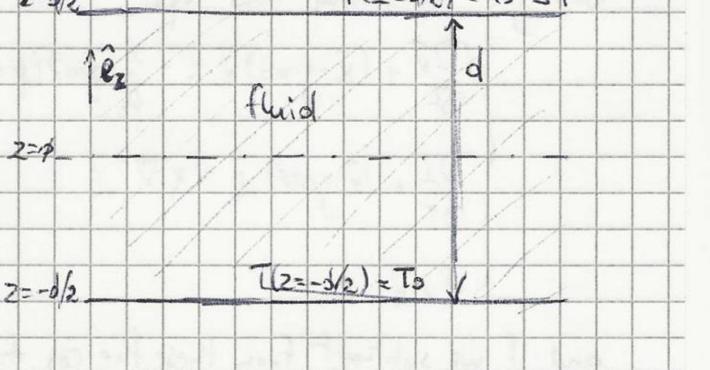
Rayleigh-Bénard instability

We present here the analysis of a steady-state convective motion setting in when a vertical temperature gradient is created across a planar layer of fluid due to the heating from below, i.e. the bottom boundary is set to a constant T_0 and the top one is at $T_0 - \Delta T$.

Below a certain limit, as we have seen, there is a stable situation where the heat transfer occurs by conduction only, so the fluid is at rest ($\bar{v} = \phi$ $\forall \bar{x} \in$ domain) and the equations describing the system, in the Boussinesq approximation, are



$$\begin{cases} -\frac{1}{\rho_0}\,\mathrm{grad}\, p' - \beta T' \bar{g} = \phi \\ \chi \frac{\partial^2 T}{\partial z^2} = \phi \end{cases}$$

where we recall that $T(z) = T_0 + T'(z)$

$$p(z) = p_0(z) + p'(z)$$

with T', p' perturbations associated to conduction.

The boundaries in $z = \pm d/2$ can be rigid walls (hence no-slip b.c. apply) or interfaces to other fluids (e.g. a heavier fluid on the bottom and a lighter one, like air, on top); as long as we are in a conduction regime there is no difference, since the velocity field is null everywhere. We also assumed (and it is coherent with the overall picture) that T, p only depend on z.

By integrating the heat eq. we get

$T(z) = Az + B$ and applying the b.c.

$$\left.\begin{aligned} T(-d/2) &= T_0 = -Ad/2 + B \\ T(d/2) &= T_0 - \Delta T = Ad/2 + B \end{aligned}\right\} \begin{aligned} &\text{summing} \Rightarrow 2T_0 - \Delta T = 2B \Rightarrow \boxed{B = T_0 - \Delta T/2} \\ &\text{subtracting} \Rightarrow \Delta T = -Ad \Rightarrow \boxed{A = -\Delta T/d} \end{aligned}$$

hence $T(z) = -\Delta T\, z/d + T_0 - \Delta T/2$ or better

$$T(z) = T_0 - \frac{\Delta T}{d}(z + d/2) = T_0 - \Gamma(z + d/2)$$

with $\Gamma \doteq \Delta T/d$ yielding a quantitative indication of the temperature gradient, in general, and here (since the $T(z)$ trend is linear) exactly $\Gamma = -\partial T/\partial z$ $\forall z$ ($p(z)$ can be obtained now that we have T, too).

Now we consider the insurgence of a convective motion due to the T gradient; we express it as a further perturbation $\bar{v}''$, T'', p'' of the velocity, temperature, pressure fields so that we write the complete fields as

$$\begin{cases} \underset{\sim}{\bar{v}}(\bar{x},t) = \phi + \bar{v}''(\bar{x},t) \\ \underset{\sim}{T}(\bar{x},t) = T(z) + T''(\bar{x},t) \\ \underset{\sim}{p}(\bar{x},t) = p(z) + p''(x,t) \end{cases}$$

where $T(z)$, $p(z)$ are the expressions obtained in the case of pure conduction

obeying the Boussinesq flow eqs.

$$\begin{cases} \frac{\partial \underset{\sim}{\bar{v}}}{\partial t} + (\underset{\sim}{\bar{v}} \cdot \text{grad})\underset{\sim}{\bar{v}} = -\frac{1}{\rho_0}\text{grad}(p' + p'') - \beta(T' + T'')\bar{g} + \nu\nabla^2\underset{\sim}{\bar{v}} \\ \frac{\partial \underset{\sim}{T}}{\partial t} + (\underset{\sim}{\bar{v}} \cdot \text{grad})\underset{\sim}{T} = \chi\nabla^2\underset{\sim}{T} \\ \text{div}\,\underset{\sim}{\bar{v}} = \phi \end{cases}$$

and if we subtract from these the eqs. for the conduction case we get the eqs. for the convective perturbation:

$$\begin{cases} \frac{\partial \bar{v}''}{\partial t} + (\bar{v}'' \cdot \text{grad})\bar{v}'' = -\frac{1}{\rho_0}\text{grad}\,p'' - \beta T''\bar{g} + \nu\nabla^2\bar{v}'' \\ \frac{\partial T''}{\partial t} + \underbrace{(\bar{v}'' \cdot \text{grad})\left(T_0 - \Gamma(z + d/2) + T''\right)}_{\hookrightarrow\, = -\Gamma v_z'' + (\bar{v}'' \cdot \text{grad})T''} = \chi\nabla^2 T'' \\ \text{div}\,\bar{v}'' = \phi \end{cases}$$

If we consider small perturbations we can linearize these eqs. by cutting off all terms that are of higher infinitesimal order than 1 (i.e. the $(\bar{v}'' \cdot \text{grad})$-type terms):

$$\begin{cases} \frac{\partial \bar{v}''}{\partial t} = -\frac{1}{\rho_0}\text{grad}\,p'' - \beta T''\bar{g} + \nu\nabla^2\bar{v}'' \\ \frac{\partial T''}{\partial t} - v_z''\Gamma = \chi\nabla^2 T'' \\ \text{div}\,\bar{v}'' = \phi \end{cases}$$

The first eq. can be manipulated in such a way that the grad p'' term disappears. If we apply the Laplacian ∇^2 to the z-component of the eq. we get

$$\frac{\partial}{\partial t}\nabla^2 v_z'' = -\frac{1}{\rho_0}\nabla^2\frac{\partial p''}{\partial z} + \beta g\nabla^2 T'' + \nu\nabla^4 v_z'' \quad (*) \quad \left(\nabla^4 = \nabla^2(\nabla^2)\right)$$

while if we apply the divergence to the whole eq. we get

$$\frac{\partial}{\partial t}\underbrace{\text{div}\,\bar{v}''}_{\phi} = -\frac{1}{\rho_0}\nabla^2 p'' + \beta g\frac{\partial T''}{\partial z} + \nu\nabla^2\underbrace{(\text{div}\,\bar{v}'')}_{\phi}$$

i.e. $-\frac{1}{\rho_0}\nabla^2 p'' = -\beta g\frac{\partial T''}{\partial z}$ and applying $\partial/\partial z$ $\quad -\frac{1}{\rho_0}\nabla^2\frac{\partial p''}{\partial z} = -\beta g\frac{\partial^2 T''}{\partial z^2}$

which we plug into (*), yielding

$$\frac{\partial}{\partial t}\nabla^2 v_z'' = \beta g\left(\nabla^2 T'' - \frac{\partial^2 T''}{\partial z^2}\right) + \nu\nabla^4 v_z'' ;$$

now we define $\nabla_H^2 \doteq \nabla^2 - \partial^2/\partial z^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ horizontal Laplacian operator,

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\nabla^2 v_z'' = \beta g \nabla_H^2 T'' + \nu \nabla^4 v_z''}$$

In order to express the b.c. for the perturbation we need now to decide explicitly on the type of the interface since $\vec{v}'' \neq \phi$. We shall consider **rigid walls and no-slip b.c.**, i.e.

$T''(x, y, z = \pm d/2) = \phi$ (already set from the beginning due to Dirichlet b.c.)

$\vec{v}''(x, y, z = \pm d/2) = \phi$

but we remark the fact that $\operatorname{div}\vec{v}'' = \phi \ \forall x, y$ and $v_x = v_y = \phi \ \forall x, y$ in $z = \pm d/2$

$\Rightarrow \operatorname{div}\vec{v}''\big|_{z=\pm d/2} = \partial v_z''/\partial z\big| = \phi$;

we can thus summarize the b.c. as $T'' = v_x'' = v_y'' = v_z'' = \partial v_z''/\partial z = \phi$ in $z = \pm d/2$.

Before we make a guess about the solution, let us normalize the eqs. to a dimensionless form. We can get a velocity scale by comparing the advective term and the $\nabla^2 T''$ term of the heat eq.:

$$v_z'' \Gamma \sim \chi \nabla^2 T''$$

$$v_z'' \Delta T/d \sim \chi \Delta T/d^2 \Rightarrow v_z'' \sim \chi/d$$

and a time scale is also similarly obtained:

$$\partial T''/\partial t \sim \chi \nabla^2 T''$$

$$\Delta T/\tau \sim \chi \Delta T/d^2 \Rightarrow \tau \sim d^2/\chi$$

so that we can introduce the rescaled, dimensionless variables

$$w = v_z''/(\chi/d)$$

$$t_p = t/(d^2/\chi)$$

$$T_p = T''/\Delta T$$

$$\vec{x}_p = \vec{x}/d$$

and differential operators as well

$\partial/\partial x_{ip} = d\,\partial/\partial x_i$; $\nabla_p = d\cdot\nabla$; $\nabla_p^2 = d^2\nabla^2$; $\nabla_{H_p}^2 = d\,\nabla_H^2$.

In the following we omit all "p" subscripts for the sake of simplicity – let us remember we are always dealing with rescaled quantities. Going on with the rescaling procedure we have:

$\odot$ $\frac{\chi}{d^2}\frac{1}{d^2}\frac{\chi}{d}\frac{\partial}{\partial t}\nabla^2 w = \beta g \frac{1}{d^2}\Delta T \nabla_H^2 T + \frac{\nu}{d^2}\frac{\chi}{d}\nabla^4 w$

i.e. $\frac{\chi^2}{d^3}\frac{\partial}{\partial t}\nabla^2 w = \beta g \Delta T \nabla_H^2 T + \frac{\nu \chi}{d^3}\nabla^4 w$ and multiplying both sides by d^3/χ^2, we get

$$\frac{\partial}{\partial t}\nabla^2 w = \left(\frac{\beta g \Delta T d^3}{\chi^2}\right)\nabla_H^2 T + \left(\frac{\nu}{\chi}\right)\nabla^4 w \quad \frac{\nu}{\chi}\to Pr \qquad \frac{\beta g \Delta T d^3}{\chi^2} = \frac{\beta g \Delta T d^3}{\nu\chi}\frac{\nu}{\chi} = Ra \cdot Pr$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\nabla^2 w = Pr\left(Ra \nabla_H^2 T + \nabla^4 w\right)} \quad \text{or} \quad \boxed{\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = Ra \nabla_H^2 T}$$

$\odot$ $\frac{\chi}{d^2}\Delta T \frac{\partial T}{\partial t} - \frac{\chi}{d}\Gamma w = \frac{\chi}{d^2}\Delta T \nabla^2 T$ and multiplying both sides by $d^2/\chi\Delta T$, we get

$\frac{\partial T}{\partial t} = \frac{d^2}{\chi \Delta T}\frac{\chi}{d}\Gamma w + \nabla^2 T$ and since $\Gamma = \Delta T/d$,

$$\boxed{\frac{\partial T}{\partial t} = \nabla^2 T + w} \quad \text{or} \quad \boxed{\left(\frac{\partial}{\partial t} - \nabla^2\right) T = w}$$

$\odot$ The rescaled b.c. are simply $\boxed{T = w = \partial w/\partial z = \phi \text{ in } z = \pm 1/2}$.

Summarizing, the perturbation problem is

$$\begin{cases} \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = Ra \nabla_H^2 T \\ \left(\frac{\partial}{\partial t} - \nabla^2\right) T = w \\ T = w = \partial w/\partial z = \phi \text{ in } z = \pm 1/2 \end{cases}$$

Let us now make a guess about the form of the solution. First of all, we assume the separation of variables in such a way we have the product between a function of z alone and a function of the other variables, say

$$w(\bar{x}, t) = \hat{w}(z)\exp(ilx + imy + \sigma t) = \hat{w}(z)\exp(i\bar{k}\cdot\bar{x} + \sigma t)$$

$$T(\bar{x}, t) = \hat{T}(z)\exp(ilx + imy + \sigma t) = \hat{T}(z)\exp(i\bar{k}\cdot\bar{x} + \sigma t)$$

with $\bar{k} = (l, m, \phi)$ wavevector of magnitude $k = (l^2 + m^2)^{1/2}$.

We have then to consider which constraints may occur for l, m, σ.

<u>l and m are real</u>: This is required to have a finite solution for $x, y \to \pm\infty$, otherwise an imaginary part of l, m would yield a real exponential (possibly diverging for $x, y \to \pm\infty$).

$\sigma = \sigma_{Re} + i\sigma_{Im} \in \mathbb{C}$ is in general acceptable; the imaginary part $\sigma_{Im} \neq \phi$ yields an imaginary exponential function in both space and time, that is to say a wave, while the real part σ_{Re} yields a real exponential function of time; we can thus distinguish the cases

$\sigma_{Re} > \phi$ perturbation growth

$\sigma_{Re} < \phi$ perturbation damping

$\sigma_{Re} = \phi$ a limit case of <u>marginal stability</u>, i.e. we either observe a steady state if $\sigma_{Im} = \phi$ or steady wave-like perturbations (they neither grow nor damp, but persist at constant amplitude) if $\sigma_{Im} \neq \phi$.

<u>Plugging</u> the guess solutions w, T into the system's eqs., since

$$\frac{\partial^2 w}{\partial x^2} = -l^2 \hat{w}(z)\exp(\dots), \quad \frac{\partial^2 w}{\partial y^2} = -m^2 \hat{w}(z)\exp(\dots), \quad \frac{\partial^2 w}{\partial z^2} = \frac{d^2\hat{w}(z)}{dz^2}\exp(\dots)$$

and similarly for T, we obtain simpler eqs. for $\hat{w}, \hat{T}$ where the laplacian operators ∇^2, ∇^2_H take on simplified forms: $\nabla^2 \to -l^2 - m^2 + d^2/dz^2 = -K^2 + d^2/dz^2$; $\nabla^2_H \to -K^2$, and similarly $\partial/\partial t \to \sigma$ $\Rightarrow$

$$\begin{cases} \left(\frac{1}{P_r}\sigma + K^2 - \frac{d^2}{dz^2}\right)\left(K^2 - \frac{d^2}{dz^2}\right)\hat{w}(z) = Ra\, K^2 \hat{T}(z) \\ \left(\sigma + K^2 - \frac{d^2}{dz^2}\right)\hat{T}(z) = \hat{w}(z) \\ \hat{w} = d\hat{w}/dz = \hat{T} = \phi \quad \text{in } z = \pm 1/2 \end{cases}$$

It can be verified* that for $Ra > \phi$, σ takes on purely real values, which means there is no wave propagation in this problem; we can only have growing, damped or steady-state solutions. As stated in the beginning, we want indeed to investigate such marginally stable case and then set <u>$\sigma = \phi$</u>. The system eqs. reduce to

$$\begin{cases} \left(K^2 - \frac{d^2}{dz^2}\right)^2 \hat{w}(z) = Ra\, K^2 \hat{T}(z) \\ \left(K^2 - \frac{d^2}{dz^2}\right)\hat{T}(z) = \hat{w}(z) \end{cases}$$

where we can use $\hat{T}(z)$ from the first eq. to plug it into the second one, and express the b.c. $\hat{T}(\pm 1/2) = \phi$ as $(d^2/dz^2 - K^2)\hat{w}(z)\big|_{\pm 1/2} = \phi$, to obtain

$$\begin{cases} \left(K^2 - \frac{d^2}{dz^2}\right)^3 \hat{w}(z) = Ra\, K^2 \hat{w}(z) \\ \hat{w} = d\hat{w}/dz = (d^2/dz^2 - K^2)\hat{w}(z) = \phi \quad \text{in } z = \pm 1/2 \end{cases}$$

* = Check, e.g. Chandrasekhar's or Drazin's books cited at the end of this note.

We ended up having a sixth-order homogeneous differential eq., that is also an eigenvalue equation: Non-trivial solutions exist based on $\mathbf{Ra}$ and K, or, in a way, they express a relationship between admissible Ra and corresponding K (coefficients will be determined as usual by imposing the b.c.; coherently with a sixth-order eq., we have six b.c. specifications).
Since we are in the case of marginal stability, this relationship $Ra(K)$ we are going to find is a value for Ra, at given K, delimiting the range of Ra values yielding a stable system (Ra < Ra(marginal stability)) and the range leading to instability (Ra > Ra(marginal stability)). This relationship $Ra(K)$ has a minimum indicating the existence of a <u>critical value</u> Ra_{cr} corresponding to a wavevector of magnitude K_{cr} for which the instability starts; that is to say, there is a certain length scale (in the horizontal direction) corresponding to K_{cr} such that a perturbation with this scale is sustained ($Ra = Ra_{cr}$) or even amplified ($Ra > Ra_{cr}$) <u>first</u>, even if originally infinitesimal. If Ra is increased, an increasing range of K (that is, of wavelengths) becomes subjected to the possibility of $\vec{v}$, T perturbation growth.

The form of the eigenfunctions for $\hat{w}(z)$ is suggested by the symmetry of the problem in z with respect to the $z=\phi$ plane; we can then say the eigenfunctions are split into two classes, i.e. symmetric and antisymmetric functions with respect to the $z=\phi$ plane (even and odd eigenfunctions, respectively). The general solution is in principle express as a superposition of functions in the form $\quad \hat{w}(z) = \exp(\pm qz)$
with $q \in \mathbb{C}$ are the six roots of the algebraic eq. obtained by plugging in the guess solution into the differential eq.:

$$\left(K^2 - d^2/dz^2\right)^3 \hat{w}(z) = Ra K^2 \hat{w}(z) \rightsquigarrow (K^2 - q^2)^3 = Ra K^2;$$

if we define $\tau^3 \doteq Ra/K^4$, the three solutions for q^2 (which we call q_0^2, q_1^2, q_2^2, and know that $q_0^2 \in \mathbb{R}$ while $q_2^2 = q_1^{2*}$ i.e. we have a real root and two complex conjugate ones) are

$$q_0^2 = -K^2(\tau - 1)$$

$$q_{1,2}^2 = K^2\left[1 + \frac{1}{2}\tau(1 \pm i\sqrt{3})\right]$$

so that we finally get the roots, which we express as

$\pm i q_0 \doteq \pm i (q_0^2)^{1/2}$

$\pm q \doteq \pm (q_1^2)^{1/2}$

$\pm q^* = \pm (q_2^2)^{1/2} = \pm (q_1^{*2})^{1/2}$

Let us see separately how even and odd eigenfunctions look like.

A) Even solutions - We can write them down as

$$\hat{w}(z) = \alpha_0 [\exp(iq_0 z) + \exp(-iq_0 z)] + \alpha [\exp(qz) + \exp(-qz)] + \alpha^* [\exp(q^* z) + \exp(-q^* z)] =$$
$$= A_0 \cos(q_0 z) + A \cosh(qz) + A^* \cosh(q^* z)$$

with $A_0 \in \mathbb{R}$, $A, A^* \in \mathbb{C}$ constants determined using the b.c.:

$$\begin{cases} \hat{w}(\pm 1/2) = 0 & \rightsquigarrow A_0 \cos(q_0/2) + A\cosh(q/2) + A^* \cosh(q^*/2) = 0 \\ d\hat{w}/dz|_{\pm 1/2} = 0 & \rightsquigarrow -A_0 q_0 \sin(q_0/2) + Aq\sinh(q/2) + A^* q^* \sinh(q^*/2) = 0 \\ (d^2/dz^2 - k^2)\hat{w}|_{\pm 1/2} = 0 & \rightsquigarrow A_0 (q_0^2 + k^2)^2 \cos(q_0/2) + A(q^2+k^2)^2 \cosh(q/2) + A^* (q^{*2} + k^2)^2 \cosh(q^*/2) = 0 \end{cases}$$

Notice that since $\hat{w}(z)$ is even, the conditions in $z = +1/2$ and $-1/2$ yield the same constraint.

We obtain a homogeneous linear system in $\bar{A} = (A_0, A, A^*)$, $\underline{\underline{M}}\bar{A} = 0$ with

$$\underline{\underline{M}} = \begin{bmatrix} \cos(q_0/2) & \cosh(q/2) & \cosh(q^*/2) \\ -q_0 \sin(q_0/2) & q\sinh(q/2) & q^* \sinh(q^*/2) \\ -(q_0^2 + k^2)\cos(q_0/2) & (q^2+k^2)\cosh(q/2) & (q^{*2}+k^2)^2 \cosh(q^*/2) \end{bmatrix}.$$

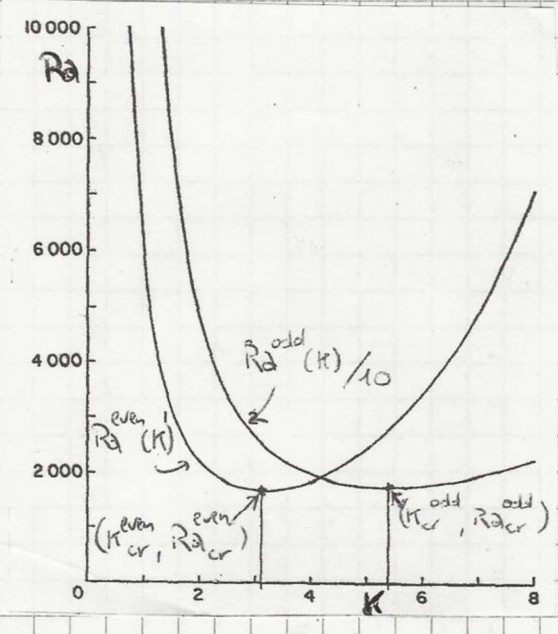
and non-trivial solutions are found, as usual, setting $\det \underline{\underline{M}} = 0$; this is a trascendental eq. whose solution can be obtained numerically; what we really want to remark here is, once again, that the solution will express a certain relationship $\mathbf{Ra}(k)$, i.e. a curve defining two regions (stability/instability) in the Ra vs k plane. The minimum of this curve is found to be at $Ra_{cr} = 1707.762$, $k_{cr} = 3.117 \rightsquigarrow \lambda_{cr} = 2\pi d / k_{cr} = 2.016\, d$

B) Odd solutions - similarly, we write them as

$$\hat{w}(z) = A_0 \sin(q_0 z) + A\sinh(qz) + A^* \sinh(q^* z)$$

and the process used to determine the actual solution is the same leading to a $Ra(k)$ curve with minimum (critical value) that is much higher ($\sim$ a factor 10 in Ra) than that for the odd functions: $Ra_{cr} = 17610.99$, $k_{cr} = 5.365 \rightsquigarrow \lambda_{cr} = 2\pi d / k_{cr} = 1.171\, d$.

We can conclude that the fundamental mode is the lowest even one, with symmetric eigenfunction with respect to $z = 0$ and no nodes in the domain $z \in (-d/2, d/2)$, which means that the vertical velocity component v_z vanishes only at the boundaries, and we can picture the flow as a single layer of <u>convection cells</u>, tiled horizontally with a periodicity λ_{cr} (the lowest odd mode would yield two layers of cells, with a node for v_z in $z=\phi$). See the pictures below.



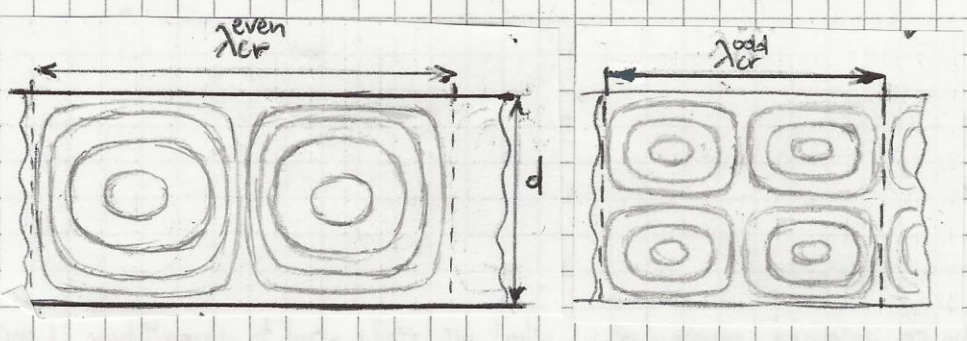
Marginal stability curves $Ra(K)$ for the lowest even and odd modes. Notice that the odd-mode curve is rescaled by a factor 10 to fit the diagram.

Critical values

[even] $K_{cr} = 3.117$; $Ra_{cr} = 1707.762$

[odd] $K_{cr} = 5.365$; $Ra_{cr} = 17610.39$

[Source: W.H. Reid, D.L. Harris, "Some Further Results on the Bénard Problem, Phys. Fluids 1, 102 (1958)]



Cell patterns for the fundamental even (left) and odd (right) convection modes

While the no-slip conditions at solid boundaries is easier to implement in a laboratory experiment, it is also worth considering a free-slip case at the surfaces $z=\pm d/2$, i.e. absence of friction (two fluid interfaces with negligible friction may fall in this description). Boundary conditions are cast as

$$v_z''(\pm d/2)=\phi \; ; \; T''(\pm d/2)=\phi \; ; \; \eta(\partial_z v_x'' + \partial_x v_z'')\Big|_{z=\pm d/2} = \eta(\partial_z v_y'' + \partial_y v_z'')\Big|_{z=\pm d/2} = \phi$$

but since $v_z''(x,y,z=\pm d/2)=\phi$ the latter condition becomes $\partial_z v_x'' = \partial_z v_y''\Big|_{z=\pm d/2} = \phi$; taking a $\partial/\partial z$ derivative of the incompressibility condition $\operatorname{div}\vec{v}''=\phi$,

$$\frac{\partial^2 v_x''}{\partial z \partial x} + \frac{\partial^2 v_y''}{\partial z \partial y} + \frac{\partial^2 v_z''}{\partial z^2} = \phi \quad \Rightarrow \quad \frac{\partial^2 v_z''}{\partial z^2}\Big|_{z=\pm d/2} = \phi$$

so that we come to the final form of the b.c., expressed for the rescaled velocity $\hat{w}$:

$\hat{w} = (d^2/dz^2 - k^2)^2\hat{w} = d^2\hat{w}/dz^2 = 0$ in $z = \pm 1/2$

but here we notice that working out the condition $(d^2/dz^2 - k^2)^2\hat{w}|_{\pm 1/2} = 0$ leads to $d^4\hat{w}/dz^4 = 0$; applying once again d^2/dz^2 we get $d^6\hat{w}/dz^6 = 0$ and iteration shows that all even derivatives of $\hat{w}$ are zero at the boundaries $z = \pm 1/2$. Hence we conclude that the eigenfunctions of the eq.

$$(k^2 - d^2/dz^2)^3\,\hat{w}(z) = Ra\,k^2\,\hat{w}(z)$$

as they vanish at the boundaries together will all their even derivatives, must take on the form

$\hat{w}(z) = A\sin(n\pi z)$ with A constant and n integer;

plugged into the differential eq., this form yields an algebraic eq.

$$(k^2 + n^2\pi^2)^3 = Ra\,k^2 \quad \Rightarrow \quad Ra = \frac{(k^2 + n^2\pi^2)^3}{k^2}$$

The critical value Ra_{cr} is a minimum of this function $Ra(k)$, therefore it must occur for the lowest mode $n=1$, and is determined setting the derivative in k (or in k^2) equal to zero:

$$\frac{d}{d(k^2)} Ra = \frac{3(k^2 + n^2\pi^2)^2}{k^2} - \frac{(k^2 + n^2\pi^2)^3}{k^4} = 0$$

$$\Rightarrow \quad 3k^2 - k^2 + n^2\pi^2 = 0 \quad \rightsquigarrow \quad k^2 = n^2\pi^2/2$$

and with $n=1$, $\quad k_{cr} = \pi/\sqrt{2} \quad (\rightsquigarrow \lambda_{cr} = 2\pi d/k_{cr} = 2\sqrt{2}\,d)$

and $\quad Ra_{cr} = (\pi^2/2 + \pi^2)^3/(\pi^2/2) = \frac{27}{4}\pi^4 = 657.511$

We could also consider a "mixed" case where the bottom boundary is solid while we have a free surface on top, and the solution is straightforward now: Notice that odd modes for the first case (solid boundary no-slip condition on both surfaces) comply with these new b.c. if we take only the lower half $(z \in (-1/2, 0))$ and rescale the results to half the layer thickness $d/2$; since $k \propto 1/d$, $Ra \propto 1/k^4$, the critical values are

$$k_{cr} = 5.365/2 = 2.682 \quad ; \quad Ra_{cr} = 17610.39/2^4 = 1100.65$$

Let us conclude with a few significant remarks.

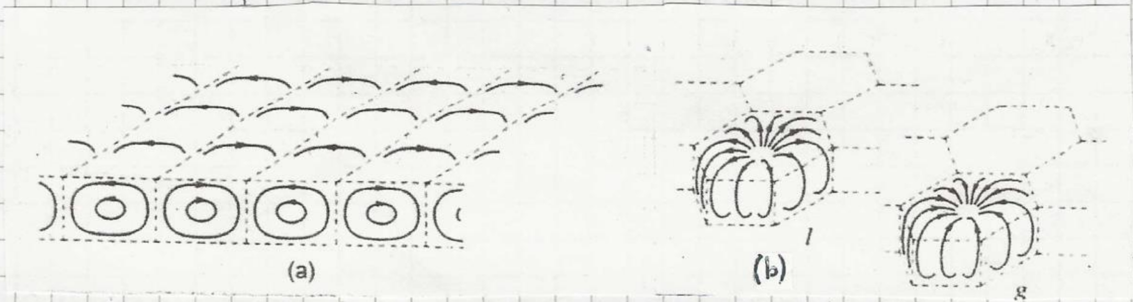
* The theory presented here is a linear stability analysis. The experimental findings are in very close agreement with such theoretical predictions which is very remarkable if we consider the degree of approximation involved in a linearization.

* We should not forget that $\kappa^2 = \ell^2 + m^2$, i.e. the wavenumber can be decomposed into the two orthogonal components in an infinite number of ways. After all, we ended up solving only the $\hat{w}(z)$ factor of the velocity field, without explicitly tackling the x- and y-dependence of v_z'' or the ~~x-~~ and y-velocity components (nor the temperature field), and finally just sketching the result graphically. For those interested in the details of the solution a deeper treatment can be found in a few books (Chandrasekhar; Drazin and Reid; Getling); we only report here that a variety of cases has been observed where the periodic structure is:

⊙ either two-dimensional (e.g. x,z-dependent, y-invariant) – so-called "2D rolls";

⊙ in the form of hexagonal or square cells in the xy plane.

The insurgence of a specific structure, or the orientation of the velocity vector in the cellular motion, is not predicted by this linear theory and depends on the features of any specific case, like the b.c. at the x- and y- edges, the value of Ra, the dependence of the properties of the fluid (e.g. viscosity) on the temperature. For instance, during the formation and growth of a crystal we can observe the same structure occurring in different areas, but laid out along non-parallel axes; the same can happen with cellular convection and non-parallel rolls in different subdomains.



(a) 2D rolls; (b) Hexagonal cells, with opposite flow orientation (ℓ- and g-type).
Source: Getling (see below).

Some useful sources

S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability", OUP (1961) or Dover (1981).

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