Solid continuous media - Elasticity

In solids, shear stresses are opposed by internal stresses that make equilibrium still possible, up to a certain limit (the yield point, beyond which the material enters a plastic regime). Therefore solids have a shape (which differentiates them with respect to fluids) within this limit, that is, within the elastic regime.

In order to describe elastic behaviour, we shall use a phenomenological approach coupling the basic law of linear elasticity, i.e. Hooke's law, with four phenomenological parameters describing solid deformation. This will allow us to build a CONSTITUTIVE EQUATION between the stress tensor and the and the strain (deformation) tensor. In this respect, let us recall that we did something similar with fluids, building a relation between stress tensor $\underline{\underline{\sigma}}$ and velocity gradient tensor $\underline{\underline{\Pi}}$; also remember that $\underline{\underline{\Pi}}$'s symmetric part $\frac{\underline{\underline{\Pi}} + \underline{\underline{\Pi}}^T}{2}$ is alternatively called strain rate tensor $\underline{\underline{\dot{E}}}$ (whose multiplication by dt yields an infinitesimal deformation indeed); the antisymmetric part is simply a rotation. The four parameters are:

- Young's modulus (or first modulus of elasticity) E;
- Bulk modulus (or modulus of compressibility) K;
- Poisson's ratio ν;
- Shear modulus (or modulus of rigidity) G.

The constitutive relation will highlight the fact that only two parameters are necessary (hence the definition of the so-called Lamé coefficients) and that we can derive relations between the four we have previously identified.

Hence we shall first obtain the stress tensor in a static condition and analyze its symmetric (lithostatic) and deviatoric part; finally we shall move to the dynamics of solids, stating the continuity equation and the Navier equation, i.e. the law describing the first Euler equation for solid continua.

Phenomenology of elasticity

Let us consider a solid bar. At rest, in the absence of external forces $\bar{F}$, the bar has length l_0 and cross section S_0. We now subject it to a force with an isothermal process, i.e. where $\bar{F}$ varies very slowly so that under the application of a small force the bar attains a new equilibrium.

We shall consider a pulling force (tension) along the axis of the bar and observe the deformation behaviour looking at two diagrams (Figs. A and B): F vs δl and and $\sigma \doteq F/S$ vs $\varepsilon \doteq \delta l / l$ where we intuitively expect that the force causes an elongation δl (as well as a variation of the cross section S).

Both diagrams show that initially the $F(\delta l)$ and $\sigma(\varepsilon)$ functions are linear up to a point A, i.e. the ELASTIC LIMIT: between O and A if the force (stress) increases, so does proportionally δl (ε) and when F decreases, then δl decreases until we revert to l_0, S_0 at $F = \emptyset$. The work $\int \bar{F} \cdot \bar{dl} = \int F dl$ done between two points on the straight line $F(\delta l)$ depends only on the extremal points and is given back when F goes back to the initial point. This is the regime of elastic behaviour and it is a reversible process (also thermodynamically).

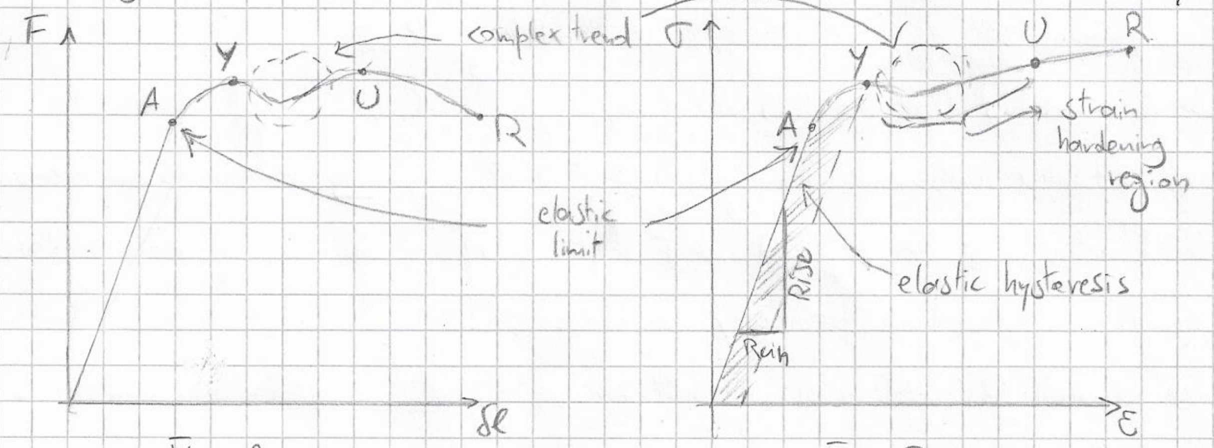
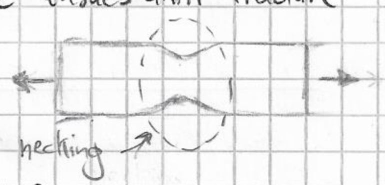


Fig. A

Fig. B

If we go further up with the force, but stay below the point Y (yield strength or stress σ_s), and then go back reducing the force, the curve is still a straigth line but along a different path, so that a permanent deformation remains even at $F = \emptyset$. The dashed area in Fig. B is the work dissipated as heat and not given back (elastic hysteresis).

Beyond Y the $F(\delta l)$ / $\sigma(\varepsilon)$ curve is complicated, and we can observe elongation even if the force is decreasing in Fig. A, one must consider anyway that with elongation, the cross section S decreases so that $\sigma = F/S$ may be going up or down even with decreasing force. In this region the material behaves like a viscous fluid, i.e. with a plastic behaviour, and reaches point U (ultimate strength point), corresponding to the maximum stress (ultimate tensile strength). This is the end of the strain hardening region, and a substantial necking (localized strain) is developed so that a reversal (drop) of the $F(\delta \ell)$ curve ensues until fracture (point R) happens. Different fracture behaviours are observed:



⊙ DUCTILE material – Y and U are far

⊙ BRITTLE material – Y and U are close (fracture before observing much elongation).

We want to examine essentially the elastic behaviour, where σ vs ε (better than F vs $\delta \ell$) is a linear function. The hypotheses we assume are:

- REVERSIBLE transformations (quasi-static processes),
- ISOTROPIC and HOMOGENEOUS materials,
- INFINITESIMAL deformations (necessary to stay within the elastic limit.

Equilibrium can be attained, or we consider either very slow or very fast processes, i.e.

⊙ reversible processes (isothermal and quasi-static transformations) or

⊙ undulatory elastic reversible processes (i.e. either at extremely high frequency such that they result isothermal as in a quasi-static process, or at frequency low enough to neglect heat conduction, such as seismic phenomena, and therefore adiabatic, which together with reversibility results in having isentropic processes).

Note: We considered a tensile stress in the example above; similar considerations would hold in case of compression.

Laws of linear elasticity

Hooke's law

Hooke's law simply states the linear relation between the stress σ and the relative deformation ε, which holds within the elastic limit. Within this range we can add the linear superposition property of Hooke's law: The effect of N stresses is the sum of each deformation ε due to each individual stress σ (where we must consider a vector sum of stresses, of course).

Axial elasticity - Young's modulus

For tension or compression, Hooke's law is expressed as

$$\frac{1}{E}\,\frac{F}{S} = \frac{\delta \ell}{\ell}, \quad \text{i.e.} \quad \sigma = E\varepsilon \text{ or } \sigma = E\varepsilon_\ell, \quad \varepsilon_\ell \text{ linear deformation}$$

and in particular $\varepsilon_\ell = \delta\ell/\ell$ is the relative elongation or contraction, and E = Young's modulus (or elastic modulus).

Note that $E > 0$ and dimensionally $[E] = [F/L^2]$ (a stress). Furthermore, E only depends on the physical status of the material (not on its geometry).

Volume elasticity - Bulk modulus

If the material sample is subjected to, say, a compression, its volume V varies by a quantity δV. If the action and strain are small we have a reversible process and by linearity (and linear superposition) $\delta V/V \propto$ pressure. Since we consider isotropic materials, the relative variation is equal in all directions, i.e. we get an isotropic contraction (or expansion, if we pulled).

Let us consider stress and strain along a certain single direction of an edge ℓ, then the linear strain is $\varepsilon_\ell = \delta\ell/\ell \propto$ pressure and the volume elasticity equation holds:

$$\varepsilon_\ell = \frac{\delta\ell}{\ell} = -\frac{1}{3K}p \qquad \text{with } K = \text{bulk modulus (or compressibility modulus)}$$

The minus sign is used so that $K > 0$, since V decreases under compression (positive p) and the factor 3 is used such that we can write

$$\varepsilon_V = \frac{\delta V}{V} = -\frac{1}{K}p \qquad \text{with } \varepsilon_V \text{ the 3-dimensional volume deformation}$$

We can prove this last equality considering a parallelepiped with edges ℓ_1, ℓ_2, ℓ_3

subjected to an isotropic pressure p stress so that $\delta l_i / l_i = -\frac{1}{3K} p \quad \forall i = 1, 2, 3$.

Hence
$$l_i' = l_i (1 + \delta l_i / l_i) = l_i (1 - p/3K)$$
$$\Rightarrow \quad V' = \prod_{i=1}^{3} l_i' = \prod_{i=1}^{3} l_i (1 - p/3K) \text{; discarding all higher order } \infty \text{esimals,}$$
$$V' = l_1 l_2 l_3 \left(1 - p/3K - p/3K - p/3K\right) = V\left(1 - \frac{1}{K} p\right)$$
$$\Rightarrow \varepsilon_V = \delta V / V = (V' - V)/V = -\frac{1}{K} p \quad \text{as we stated above.}$$

Notes: $\alpha_p \doteq 1/K$ is also defined and called isothermal compressibility coefficient; dimensionally, $[K] = [F/L^2] = [\text{stress}]$.

Poisson's ratio

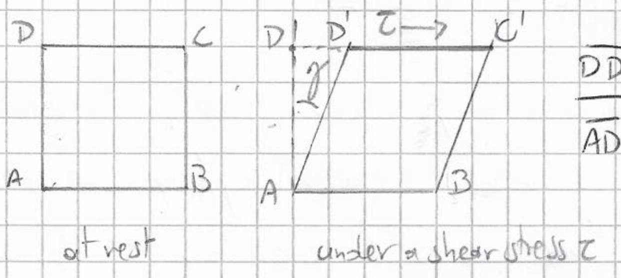
If we pull a bar, as it elongates it also shrinks in cross section S, that is, its transverse linear dimensions decrease; in isotropic materials, the relative transverse linear variation $\delta b / b$ is equal in all directions.

We call Poisson's ratio ν the ratio between transverse and longitudinal relative variations (with a minus sign):
$$\frac{\delta b}{b} = -\nu \frac{\delta l}{l} = -\frac{\nu}{E} \sigma \quad ;$$

ν is dimensionless and thanks to the minus sign $\nu > 0$ for "normal" materials ($\nu < 0$ for special cases called auxetic materials), where we observe transverse contraction for longitudinal elongation and vice versa. ν depends only on the material status.

Shear modulus

It is a measure of the elastic shear stiffness of a material. Consider a sample as in the figure; in a two-dimensional cross section, a rectangular shape $ABCD$ at rest. When a shear stress τ is applied on the top surface (DC edge) while the bottom (AB edge) is held in place, the shear-strained sample attains a new shape $ABC'D'$ we can measure by



$$\frac{\overline{DD'}}{\overline{AD}} = \tan\gamma \simeq \gamma \text{ (for } \infty\text{esimal strain)}$$

Hooke's law for elastic behaviour (linear relation between stress and strain) tells us

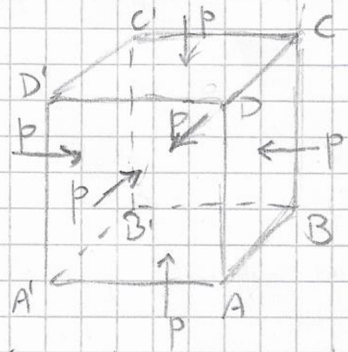
that $\gamma = \frac{1}{G}\tau$ with G = shear modulus (or modulus of rigidity)

Again, dimensionally $[G] = [F/L^2] = [\text{stress}]$ and must be a material (not geometrical) property, and $G > \phi$.

Relations between E, K, ν, γ and constraints

As previously mentioned, these properties are not independent.

First, let us consider a cube subjected to a uniform pressure on all faces:



The deformation of the edge AA' is, by linear superposition,

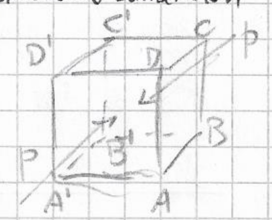
$$\frac{\delta l}{l} = \frac{\delta l_1}{l_1} + \frac{\delta l_2}{l_2} + \frac{\delta l_3}{l_3}$$ where

$\delta l / l$ is the relative length deformation due to volume elasticity under isotropic p, and is equivalently given as a superposition of effects:

- $\delta l_1 / l_1$ relative axial deformation due to pressure on the faces $ABCD$, $A'B'C'D'$,

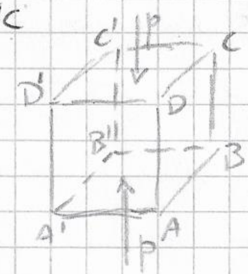
$\Rightarrow \frac{\delta l_1}{l_1} = -\frac{1}{E}p$ (with a minus sign since for positive p we have compression);

- $\delta l_2 / l_2$ relative deformation of the edge AA' considered as a transverse dimension with respect to the pressure applied to faces $AA'D'D$, $BB'C'C$



$\Rightarrow \frac{\delta l_2}{l_2} = -\nu \frac{\delta l_1}{l_1} = \frac{\nu}{E}p$;

- $\delta l_3 / l_3$ similarly to $\delta l_2 / l_2$, due to pressure on $AA'B'B$, $DD'C'C$



$\Rightarrow \frac{\delta l_3}{l_3} = -\nu \frac{\delta l_1}{l_1} = \frac{\nu}{E}p$;

Summarizing,

$$\frac{\delta l}{l} = -\frac{1}{3K}p = -\frac{1}{E}p + \frac{\nu}{E}p + \frac{\nu}{E}p = (-1+2\nu)\frac{p}{E} \Rightarrow$$

$$\boxed{K = \frac{E}{3(1-2\nu)}}$$

which actually does not depend on the geometry and is hence of general validity (beyond the cubic example).

Second, one can prove there is a relation linking the shear modulus G to the other parameters,
$$\boxed{G = \frac{E}{2(1+\nu)}}$$

Now notice that these two constraints not only reduce the set of free parameters from four down to two, but also yield constraints on their magnitude. Since $K > 0$, $E > 0$, the relation $K = E/3(1-2\nu)$ imposes

$1-2\nu > 0 \Rightarrow \underline{\nu < 1/2}$;

we said that for "normal" materials, $\nu \geq 0$, but that negative ν (auxetic) ones exist; ν has a lower limit nonetheless, since the second relation $G = E/2(1+\nu)$ now imposes

$1+\nu > 0 \Rightarrow \underline{\underline{\nu > -1}}$

so that, all in all, $\boxed{-1 < \nu < 1/2 \quad (\text{with } E, K, G > 0).}$

Typical values for ν are in the range $0.2 \div 0.4$, so that E and K are typically of the same order of magnitude (tens to few hundred GPa for rocks and various metallic materials).

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A nice application of our considerations on elasticity is the estimate of the maximum height of mountains. Indeed the maximum pressure at the bottom of a mountain must be within the yield strength σ_s, otherwise plastic behaviour would occur and the rock would start to "melt down" by flowing like a viscous fluid under the applied stress. The lithostatic stress, i.e. the stress in the solid material in a static condition, yields a vertical stress $p = \rho g h$ (we shall discuss later the lithostatic stress tensor); the typical yield strength σ_s for rocks is slightly below 300 MPa, with an average density $\rho \approx 3 \cdot 10^3\ Kg/m^3$; requesting

$$\rho g H_{max} \leq \sigma_s \quad \text{yields} \quad H_{max} \leq \sigma_s/\rho g$$

and we can make an estimate for Earth, with $g_E = 9.81\ m/s^2$, and Mars, with $g_M = 3.71\ m/s^2$ ($g_E/g_M \simeq 2.65$), yielding $H_{max}(\text{Earth}) \simeq 10$ Km, $H_{max}(\text{Mars}) \simeq 27$ Km. Even within this rough estimate numbers are consistent with the respective highest mountains: Everest's 8,85 Km and Olympus' 22 Km.

Strain and stress tensors - Constitutive equation

First of all, let us recall the concept of strain tensor. A quick and simple conceptual approach is starting from the strain rate tensor $\underline{\underline{\dot{\varepsilon}}} = \underline{\underline{\dot{\Pi}}}^S + \underline{\underline{\dot{\Pi}}}^I$ and observing that a multiplication by dt gets us terms in the form $\partial_j v_i dt \sim \partial_j du_i$, i.e. an infinitesimal deformation (strain) gradient as du_i is an infinitesimal displacement; therefore all the considerations elaborated about $\underline{\underline{\Pi}}$ still hold.

But let us work this out more properly from scratch. We shall call DISPLACEMENT the vector $\bar{u}$ representing the change in position of a point in the medium due to the material's deformation: $\bar{u} = \bar{x}' - \bar{x}$ or in components $u_i = x_i' - x_i$.

If we take two very close points Q and P, such that $\overline{P-Q}$ is an infinitesimal distance, in the deformation, during time, they displace to Q' and P', respectively. So we can say that, to first order,

$$u_i(P,t) = u_i(Q,t) + (\overline{P-Q})_j \frac{\partial}{\partial x_j} u_i(\bar{x},t)\Big|_{\bar{x}=\bar{x}(Q,t)} =$$

$$= u_i(Q,t) + (\overline{P-Q}) \cdot \text{grad}\, u_i(\bar{x}(Q,t))$$

and by defining $\Pi_{ij} \doteq \partial u_i / \partial x_j$ DISPLACEMENT GRADIENT TENSOR

we get

$$u_i(P,t) = u_i(Q,t) + (\overline{P-Q})_j \Pi_{ij}$$

or equivalently

$$(\overline{P'-P})_i = (\overline{Q'-Q})_i + (\overline{P-Q})_j \Pi_{ij}.$$

We can perform some useful manipulation here. Let us consider

$$(\overline{P'-Q'})_i = (\overline{P'-P})_i + (\overline{P-Q})_i + (\overline{Q-Q'}) = \text{using the expressions above}$$

$$= \cancel{(\overline{Q'-Q})_i} + (\overline{P-Q})_j \Pi_{ij} + \underbrace{(\overline{P-Q})_i}_{=(\overline{P-Q})_j \delta_{ij}} - \cancel{(\overline{Q'-Q})}$$

$$\Rightarrow (\overline{P'-Q'})_i = (\overline{P-Q})_j (\delta_{ij} + \Pi_{ij}(Q))$$

or $(\overline{P'-Q'}) = (\underline{\underline{1}} + \underline{\underline{\Pi}})(\overline{P-Q})$ which shows the time evolution of $(\overline{P-Q})$ due to the deformation.

As usual since $\underline{\underline{\Pi}}$ is a second-order tensor, a unique decomposition exists

$$\underline{\underline{\Pi}} = \underline{\underline{\Pi}}^S + \underline{\underline{\Pi}}^A + \underline{\underline{\Pi}}^I$$

and explicitly

$$\Pi_{ij} = \frac{1}{2}\left[\partial_j u_i + \partial_i u_j - \frac{2}{3}\partial_\ell u_\ell \delta_{ij}\right] + \frac{1}{2}\left[\partial_j u_i - \partial_i u_j\right] + \frac{1}{3}\partial_\ell u_\ell \delta_{ij},$$

so we define $\underline{\underline{\varepsilon}} = \underline{\underline{\Pi}}^S + \underline{\underline{\Pi}}^I$ INFINITESIMAL STRAIN TENSOR

with $\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$

while $\underline{\underline{\Pi}}^A = \underline{\underline{\omega}}$ infinitesimal rotation tensor is something we do not care about since it describes a rigid rotation (and we study elastic deformation only).

Remember that all of the above is valid to a first-order infinitesimal approximation, that is, for infinitesimal deformations. If we had to accept finite deformation, the strain tensor would include a higher-order term and read $\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j + \partial_i u_\ell \partial_j u_\ell)$.

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Now we can the hypotheses for a constitutive relation:

- There is no intrinsic (spin-like) angular momentum in the medium.
- By Hooke's law, stresses are linearly proportional to deformations and superposition holds.
- The medium is isotropic (which is a more restrictive requirement of the more general <u>PRINCIPLE OF MATERIAL-FRAME INDIFFERENCE</u>, stating that the properties of the material are the same whatever coordinate system is used to observe them; isotropy requires same properties in any direction).

Notice that the requests are the same ones we used for the constitutive relation $\underline{\underline{\sigma}}' - \underline{\underline{\Pi}}$ of Newtonian fluids: the first one (no intrinsic spin) leads to a symmetric $\underline{\underline{\sigma}}$, the second one to a relationship of the kind $\sigma_{ij} = C_{ijhk}\varepsilon_{hk}$, the third one requires coefficients in the constitutive eq. that do not depend on the coordinate system, i.e. C_{ijhk} isotropic fourth-order tensor; all in all, we must conclude that

$$\underline{\underline{\sigma}} = A\underline{\underline{\varepsilon}}^S + B\underline{\underline{\varepsilon}}^I \qquad \text{with } A, B \text{ scalar coefficients}$$

and since a traceless symmetric tensor x scalar yields another traceless symmetric tensor (and so for the isotropic part),

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^S + \underline{\underline{\sigma}}^I = A\underline{\underline{\varepsilon}}^S + B\underline{\underline{\varepsilon}}^I \qquad \text{with } \underline{\underline{\sigma}}^S = A\underline{\underline{\varepsilon}}^S,\ \underline{\underline{\sigma}}^I = B\underline{\underline{\varepsilon}}^I,$$

and, in a fully explicit fashion,

$$\sigma_{ij} = \frac{1}{2}A\left(\partial_j u_i + \partial_i u_j + \frac{2}{3}\operatorname{div}\vec{u}\,\delta_{ij}\right) + \frac{1}{3}B\operatorname{div}\vec{u}\,\delta_{ij}$$

One can prove (we do not do it here) that

$$A = 2G \quad ; \quad B = 3K$$

$$\Rightarrow \boxed{\underline{\underline{\sigma}} = 2G\,\underline{\underline{\varepsilon}}^S + 3K\,\underline{\underline{\varepsilon}}^I} \quad \text{or} \quad \sigma_{ij} = G(\partial_j u_i + \partial_i u_j + \tfrac{2}{3}\,\mathrm{div}\,\vec{u}\,\delta_{ij}) + K\,\mathrm{div}\,\vec{u}\,\delta_{ij}$$

This is the most sensible decomposition of $\underline{\underline{\sigma}}$, as it is the one and only geometric decomposition of the tensor; nevertheless the most common decomposition that is well established in the study of elasticity is another one, originating from the fact that

$$\underline{\underline{\varepsilon}}^I = \frac{1}{3}\mathrm{Tr}(\underline{\underline{\varepsilon}})\underline{\underline{\mathbb{1}}} \quad ; \quad \underline{\underline{\varepsilon}}^S = \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^I = \underline{\underline{\varepsilon}} - \frac{1}{3}\mathrm{Tr}(\underline{\underline{\varepsilon}})\underline{\underline{\mathbb{1}}}$$

$$\Rightarrow \underline{\underline{\sigma}} = 2G\underline{\underline{\varepsilon}} - 2G\underline{\underline{\varepsilon}}^I + 3K\underline{\underline{\varepsilon}}^I = 2G\underline{\underline{\varepsilon}} + (3K-2G)\underline{\underline{\varepsilon}}^I = 2G\underline{\underline{\varepsilon}} + \left(K - \frac{2}{3}G\right)\mathrm{Tr}(\underline{\underline{\varepsilon}})\underline{\underline{\mathbb{1}}}$$

which is then typically written down as

$$\boxed{\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}} + \lambda\,\mathrm{Tr}(\underline{\underline{\varepsilon}})\underline{\underline{\mathbb{1}}}}$$

with

$$\begin{cases} \mu \doteq G \\ \lambda \doteq K - \frac{2}{3}G \end{cases} \qquad \mu, \lambda \text{ called } \underline{\text{Lamé coefficients}}$$

Notice that in both decompositions only two coefficients are needed to express the constitutive law relating $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$. This is a consequence of the isotropy (an anisotropic material like a crystal would require four).

We can also invert the constitutive relations:

$$\underline{\underline{\varepsilon}}^S = \frac{1}{A}\underline{\underline{\sigma}}^S = \frac{1}{2G}\underline{\underline{\sigma}}^S \quad ; \quad \underline{\underline{\varepsilon}}^I = \frac{1}{B}\underline{\underline{\sigma}}^I = \frac{1}{3K}\underline{\underline{\sigma}}^I \Rightarrow \boxed{\underline{\underline{\varepsilon}} = \frac{1}{2G}\underline{\underline{\sigma}}^S + \frac{1}{3K}\underline{\underline{\sigma}}^I}$$

If we want to follow with the alternative decomposition,

$$\underline{\underline{\sigma}}^I = \frac{1}{3}\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}} \quad ; \quad \underline{\underline{\sigma}}^S = \underline{\underline{\sigma}} - \underline{\underline{\sigma}}^I = \underline{\underline{\sigma}} - \frac{1}{3}\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}}$$

$$\Rightarrow \varepsilon_{ij} = \frac{1}{A}\sigma_{ij} - \frac{1}{A}\frac{1}{3}\sigma_{\ell\ell}\delta_{ij} + \frac{1}{B}\frac{1}{3}\sigma_{\ell\ell}\delta_{ij} = \frac{1}{A}\sigma_{ij} + \left(\frac{1}{3B} - \frac{1}{3A}\right)\sigma_{\ell\ell}\delta_{ij}$$

$$\text{or } \underline{\underline{\varepsilon}} = \frac{1}{A}\underline{\underline{\sigma}} - \frac{1}{3A}\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}} + \frac{1}{3B}\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}} = \frac{1}{A}\underline{\underline{\sigma}} + \left(\frac{1}{3B} - \frac{1}{3A}\right)\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}}$$

so that finally

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2G}\underline{\underline{\sigma}} + \left(\frac{1}{9K} - \frac{1}{6G}\right)\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}}}$$

or with some algebra and the Lamé coefficients,

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2\mu}\underline{\underline{\sigma}} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}}} \quad \left(\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{\ell\ell}\delta_{ij}\right)$$

Note: We could, in a way, get straight to the Lamé-coefficient form of the constitutive eq. if we sort of diligently worked out the hypotheses in full as done for the Newtonian fluid. Indeed we have

$$\sigma_{ij} = C_{ijhk}\,\varepsilon_{hk} \;;\quad \varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)\;;$$

$$C_{ijhk} = \lambda\delta_{ij}\delta_{hk} + \mu\delta_{ih}\delta_{jk} + \eta\delta_{ik}\delta_{jh}$$

(^{s} indicates the whole symmetric part)

$$\Rightarrow \sigma_{ij} = \lambda\delta_{ij}\delta_{hk}\varepsilon_{hk} + \mu\delta_{ih}\delta_{jk}\varepsilon_{hk} + \eta\delta_{ik}\delta_{jh}\varepsilon_{hk} =$$

$$= \lambda\varepsilon_{hh}\delta_{ij} + \mu\varepsilon_{ij} + \eta\varepsilon_{ji} = \lambda\varepsilon_{hh}\delta_{ij} + \mu(\varepsilon_{ij}^{s} + \varepsilon_{ij}^{A}) + \eta(\underbrace{\varepsilon_{ji}^{s}}_{=\varepsilon_{ij}^{s}} + \underbrace{\varepsilon_{ji}^{A}}_{=-\varepsilon_{ij}^{A}}) =$$

$$= \lambda\varepsilon_{hh}\delta_{ij} + (\mu+\eta)\varepsilon_{ij}^{s} + (\mu-\eta)\varepsilon_{ij}^{A}$$

since $\varepsilon_{ij} = \varepsilon_{ij}^{s}$ only, ε_{ij}^{A} must vanish ($\Rightarrow \eta = \mu$)

$$\boxed{\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{hh}\delta_{ij}} \quad \text{or} \quad \boxed{\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}} + \lambda\,\mathrm{Tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{\mathbb{1}}}}$$

Also $\mathrm{Tr}(\underline{\underline{\sigma}}) = 2\mu\,\mathrm{Tr}(\underline{\underline{\varepsilon}}) + 3\lambda\,\mathrm{Tr}(\underline{\underline{\varepsilon}}) = (2\mu+3\lambda)\,\mathrm{Tr}(\underline{\underline{\varepsilon}}) \Rightarrow \mathrm{Tr}(\underline{\underline{\varepsilon}}) = \frac{1}{2\mu+3\lambda}\,\mathrm{Tr}\,\underline{\underline{\sigma}}$

so in the Lamé decomposition, $\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}} + \frac{\lambda}{2\mu+3\lambda}\,\mathrm{Tr}(\underline{\underline{\sigma}})\,\underline{\underline{\mathbb{1}}}$

and inverting for $\underline{\underline{\varepsilon}}$ we get $\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2\mu}\underline{\underline{\sigma}} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\,\mathrm{Tr}(\underline{\underline{\sigma}})\,\underline{\underline{\mathbb{1}}}}$ $\left(\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{ee}\delta_{ij}\right)$

A few conclusive comments.

- Elastic behaviour can also be seen as a no-memory behaviour: Phenomena do not depend on the past history of the media.
- Principle of frame indifference (objectivity): Governing laws must have the same from in any reference frame – there is no "preferred" frame such as the "inertial" frame. We typically have to <u>postulate</u> that the stress tensor is objective.

  Principle of material-frame indifference: Materials are indifferent to reference frames, i.e. the constitutive relations have the same form in whatever reference frame. Again, this can well be controversial and is simply postulated (inertial effects at the microscopic scale may break this hypothesis down; We really need to assume a continuous medium model).
- Material isotropy as assumed by our treatment of elastic solids and Newtonian fluids is a special (and very stringent) case of material-frame indifference (or material objectivity).

⊙ Knowing $A = 2G$, $B = 3K$, with some algebra we can get the full set of relations:

$$\frac{1}{E} = \frac{1}{3G} + \frac{1}{9K} \quad ; \quad \nu = \frac{1/6G - 1/9K}{1/3G + 1/9K}$$

and for the Lamé coefficients

$$\begin{cases} \mu = G \\ \lambda = K - \frac{2}{3}G \end{cases} \longrightarrow \begin{cases} \lambda = \nu E / (1+\nu)(1-2\nu) \\ \mu = E / 2(1+\nu) \end{cases} \Rightarrow \lambda + 2\mu = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

and the constraints previously given for E, K, ν, G yield constraints for the Lamé coefficients $\mu > 0$; $\lambda > -\frac{2}{3}\mu$.