Statics - The lithostatic stress tensor

We want to determine the distribution of stresses in a static (equilibrium) situation for elastic solids such as rocks (e.g. in the depths of a mountain or in the planet's crust), hence the name of lithostatic stress tensor. The founding hypotheses are:

1) Rock is elastic.

2) The system is at rest (and hence time-independent).

3) The stress exerted on a horizontal plane is due only to the weight of the overlaying column of material (overburden pressure).

4) The system is invariant under horizontal translation, hence a vertical column of rock cannot expand/contract laterally, since it is bound on all sides by rock with the exact same properties.

We can build a set of displacements and hence of stresses based on the invariances and symmetries stated above. Hypothesis 4) implies that the displacement $\vec{u}$ only has nonzero component in the vertical direction (u_z coincides with u) and only depends on the vertical coordinate, which we shall call z: $\vec{u} = u(z)\hat{e}_z$. Hence the strain tensor reads

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_z u(z) \end{pmatrix} \quad ; \quad Tr(\underline{\underline{\varepsilon}}) = \partial_z u(z) \text{ hence we get } \underline{\underline{\sigma}}:$$

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}} + \lambda\, Tr(\underline{\underline{\varepsilon}})\underline{\underline{\mathbb{1}}} = \begin{pmatrix} \lambda\partial_z u(z) & 0 & 0 \\ 0 & \lambda\partial_z u(z) & 0 \\ 0 & 0 & (2\mu+\lambda)\partial_z u(z) \end{pmatrix}$$

It is important to notice that unlike the hydrostatic equilibrium stress tensor, i.e. the stress tensor for fluids in a static situation ($\sigma_{ij} = -p\delta_{ij}$), even at equilibrium the lithostatic stress tensor is <u>NOT</u> isotropic.

Let us use hyp. 3) to determine the normal vertical stress σ_{zz}. The stress is directed along $-\hat{e}_z$, as it points downwards. We consider a uniform gravitational acceleration g and also notice that the density of rock not subjected to stresses is ρ, while compression yields $\rho+\rho'$ with a perturbation ρ' that we consider infinitesimal (we always within an elastic regime of infinitesimal variations). With these assumptions, the stress σ_{zz} exerted on a horizontal plane at a depth h within the rock is the weight of the vertical column above, with height h and density ρ:

$$\sigma_{zz} = -\rho g h \Rightarrow (2\mu+\lambda)\partial_z u(z) = -\rho g h \Rightarrow \partial_z u(z) = -\rho g h/(2\mu+\lambda)$$

So that the stress tensor reads

$$\underline{\underline{\sigma}} = \begin{pmatrix} -\frac{\lambda}{2\mu+\lambda}\rho g h & \phi & \phi \\ \phi & -\frac{\lambda}{2\mu+\lambda}\rho g h & \phi \\ \phi & \phi & -\rho g h \end{pmatrix};$$

note $\lambda = \nu E/(1+\nu)(1-2\nu)$, $\lambda + 2\mu = E(1-\nu)/(1+\nu)(1-2\nu) \Rightarrow$

equivalently

$$\underline{\underline{\sigma}} = \begin{pmatrix} -\frac{\nu}{1-\nu}\rho g h & \phi & \phi \\ \phi & -\frac{\nu}{1-\nu}\rho g h & \phi \\ \phi & \phi & -\rho g h \end{pmatrix}$$

LITHOSTATIC STRESS TENSOR

which, as we said, is not isotropic; the isotropic part is

$$\underline{\underline{\sigma}}^I = \frac{1}{3}\mathrm{Tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbb{1}}} = -\frac{1+\nu}{3(1-\nu)}\rho g h \delta_{ij}$$

(and some call this, while others $-\rho g h \delta_{ij}$, the lithostatic tensor; mind the nomenclature!)

so that the "deviatoric" stress component $\underline{\underline{\sigma}}^S$ is

$$\underline{\underline{\sigma}}^S = \underline{\underline{\sigma}} - \frac{1}{3}\mathrm{Tr}(\sigma)\underline{\underline{\mathbb{1}}} = \begin{pmatrix} \frac{1-2\nu}{3(1-\nu)}\rho g h & \phi & \phi \\ \phi & \frac{1-2\nu}{3(1-\nu)}\rho g h & \phi \\ \phi & \phi & \frac{-2+4\nu}{3(1-\nu)}\rho g h \end{pmatrix}$$

Another way to decompose $\underline{\underline{\sigma}}$ is considering a deviatoric part with respect to the isotropic "hydrostatic" component $-\rho g h \underline{\underline{\mathbb{1}}}$:

$$\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - (-\rho g h \underline{\underline{\mathbb{1}}}) = \begin{pmatrix} \frac{1-2\nu}{1-\nu}\rho g h & \phi & \phi \\ \phi & \frac{1-2\nu}{1-\nu}\rho g h & \phi \\ \phi & \phi & \phi \end{pmatrix}$$

__Note__: One must remember that this last decomposition is not a geometric one (only one can be) and therefore depends on the frame of reference (the two parts do not transform independently like $\underline{\underline{\sigma}}^S$ and $\underline{\underline{\sigma}}^I$ would do).

Lithostatic stresses on non-horizontal planes

⊙ Vertical planes

Let us consider a stress $\bar{f}$ on a vertical plane. Assume first that the plane has $\hat{e}_x$ as the unit normal vector. $\bar{f}$ only possesses normal component along $\hat{e}_x$, i.e.

$$f_x = \sigma_{xx} = -\frac{\nu}{1-\nu}\rho g h$$

$$f_y = \sigma_{xy} = \phi$$

$$f_z = \sigma_{xz} = \phi$$

and similarly for a plane with normal unit vector $\hat{e}_y$.

In general, for any vertical plane with normal unit vector $\hat{n} = n_1\hat{e}_1 + n_2\hat{e}_2$ (say we take $x=1, y=2, z=3$),

$$\left.\begin{aligned} f_1 &= \sigma_{11}n_1 + \sigma_{12}n_2 = -\frac{\nu}{1-\nu}\rho g h\, n_1 \\ f_2 &= \sigma_{21}n_1 + \sigma_{22}n_2 = -\frac{\nu}{1-\nu}\rho g h\, n_2 \\ f_3 &= \sigma_{31}n_1 + \sigma_{32}n_2 = \phi \end{aligned}\right\} \Rightarrow \bar{f} = -\frac{\nu}{1-\nu}\rho g h\,\hat{n} \text{ normal to the plane}$$

with $|\bar{f}| = \frac{\nu}{1-\nu}\rho g h \neq \rho g h$ hydrostatic stress!

⊙ Planes neither vertical nor horizontal

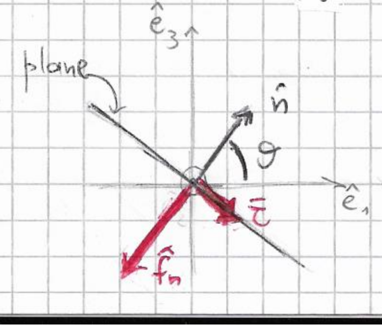
The anisotropy of the tensor $\underline{\underline{\sigma}}$ implies that on such planes the stress has both normal and tangential (shear) components. We consider again $x=1, y=2, z=3$ vertical direction, so that in this reference frame $\underline{\underline{\sigma}}$ reads

$$\underline{\underline{\sigma}} = \begin{pmatrix} \lambda_1 & \phi & \phi \\ \phi & \lambda_2 & \phi \\ \phi & \phi & \lambda_3 \end{pmatrix}$$

with λ_i eigenvalues and

$$\lambda_1 = \lambda_2 = -\frac{\nu}{1-\nu}\rho g h \neq \lambda_3 = -\rho g h$$

Our inclined plane has normal unit vector $\hat{n}$; the n_3 component and at least one among n_1, n_2 are non-zero. Let us assume for simplicity $n_1 \neq \phi$, $n_2 = \phi$ so that $\hat{n} = (n_1, \phi, n_3)$. The axis $\hat{e}_2$ belongs to this plane; also the angle between the $\hat{n}$ and $\hat{e}_1$ vectors shall be named ϑ.



Now
$$f_1 = \sigma_{1j}n_j = \lambda_1 n_1$$
$$f_2 = \sigma_{2j}n_j = \lambda_2 n_2 = \phi$$
$$f_3 = \sigma_{3j}n_j = \lambda_3 n_3$$

If $\bar{f}$ were perpendicular to the desired plane then it would be $\hat{f} // \hat{n}$ while $\hat{f}\times\hat{n} = \emptyset$; we can show this is not the case:

$$(\bar{f}\times\hat{n})_1 = f_2 n_3 - f_3 n_2 = \emptyset$$

$$(\bar{f}\times\hat{n})_2 = f_3 n_1 - f_1 n_3 = \lambda_3 n_3 n_1 - \lambda_1 n_1 n_3 = (\lambda_3 - \lambda_1) n_1 n_3 = \frac{-1+2\nu}{1-\nu}\rho g h\, n_1 n_3$$

$$(\bar{f}\times\hat{n})_3 = f_1 n_2 - f_2 n_1 = \emptyset$$

and $\bar{f}\times\hat{n} = \emptyset \iff -1+2\nu = \emptyset \rightsquigarrow \nu = 0.5$ out of the acceptable range of values for ν $\Rightarrow$ there is a shear stress

The normal stress is

$$f_n = \bar{f}\cdot\hat{n} = f_1 n_1 + f_2 n_2 + f_3 n_3 = \lambda_1 n_1^2 + \cancel{\lambda_2 n_2^2} + \lambda_3 n_3^2 = -\left(\frac{\nu}{1-\nu} n_1^2 + n_3^2\right)\rho g h$$

while the shear stress is $\bar{\tau} = \hat{n}\times(\bar{f}\times\hat{n})$; the modulus is

$$\tau = |\hat{n}\times(\bar{f}\times\hat{n})| = \left|\hat{n}\times \frac{-1+2\nu}{1-\nu} n_1 n_3 \rho g h\, \hat{e}_2\right| = \left|\frac{-1+2\nu}{1-\nu} n_1 n_3\right| \rho g h$$

and the direction is $\hat{n}\times\hat{e}_2 = -n_3\hat{e}_1 + n_1\hat{e}_3$.

Example: We have a granite massif and we want to calculate the stresses at a depth $h = 1000$ m within the rock, on a horizontal, vertical and tilted plane respectively. The inclination of the plane's normal vector is $\theta = 45°$. The granite density is $\rho = 2.6\cdot10^3$ Kg/m^3 and the Poisson's ratio is $\nu = 0.25$.

On the horizontal plane $|\sigma_{33}| = \rho g h \simeq 25.5$ MPa ($\simeq 255$ atm) the vertical (normal) stress is the only nonzero component.

On the vertical plane we have normal stress only, once more, say, along direction $\hat{e}_1$

$$\Rightarrow |\sigma_{11}| = \frac{\nu}{1-\nu}\rho g h = 8.5 \text{ MPa } (\simeq 85 \text{ atm}) \quad \text{a third of the vertical one.}$$

On the tilted plane, let us consider as above nonzero $\hat{e}_1$ component;

the normal stress is $f_n = \left|-\left(\frac{\nu}{1-\nu} n_1^2 + n^3\right)\rho g h\right|_{n_1 = n_3 = \sqrt{2}/2 \text{ with } \theta = 45°} = 17$ MPa ($\simeq 170$ atm) along $-(\hat{e}_1 + \hat{e}_3)$

while the shear stress is $\tau = \left|\frac{-1+2\nu}{1-\nu} n_1 n_3 \rho g h\right| = 8.5$ MPa ($\simeq 85$ atm) along $(-n_3\hat{e}_1 + n_1\hat{e}_3)$

or in vector form $\bar{\tau} = \frac{-1+2\nu}{1-\nu}\rho g h\, n_1 n_3(-n_3\hat{e}_1 + n_1\hat{e}_3) = (-\tau)\frac{\sqrt{2}}{2}(\hat{e}_3 - \hat{e}_1) = 6\,(\hat{e}_1 - \hat{e}_3)$ MPa

since $\frac{-1+2\nu}{1-\nu} = -\frac{2}{3}$, negative!

Dynamics - Continuity and Navier equations

Continuity equation

Let us call $\rho_e(\bar{x})$ the density of the medium at equilibrium, and $\rho(\bar{x},t)$ the density distribution associated to a state of motion, so that the density perturbation is

$$\rho'(\bar{x},t) = \rho(\bar{x},t) - \rho_e(\bar{x})$$

which we consider to be infinitesimal for infinitesimal displacement. Therefore if we want to write the continuity equation we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\rho_e+\rho') = -\mathrm{div}(\rho\bar{v}) = -\mathrm{div}[(\rho_e+\rho')\bar{v})] \simeq -\mathrm{div}(\rho_e\bar{v})$$

as $\mathrm{div}(\rho'\bar{v})$ is a higher-order infinitesimal quantity. Hence we get (as $\partial_t \rho_e = \emptyset$)

$$\boxed{\frac{\partial \rho'}{\partial t} = -\mathrm{div}(\rho_e\bar{v})}$$ CONTINUITY EQUATION

Notice that while very often we may assume ρ_e = uniform, in extreme cases like a very thick layer of material the amount of weight due to the action of gravity may introduce a density gradient.

Navier equation

The law of motion for a body is expressed through the first Euler's equation; for a continuum element of mass m,

$$\frac{D}{Dt}(m\bar{v}) = \frac{m}{\rho}\,\mathrm{div}\,\underline{\underline{\sigma}} + m\bar{f} \quad \text{(momentum balance eq.)}$$

where as usual $\frac{1}{\rho}\mathrm{div}\,\underline{\underline{\sigma}}$ = surface forces per unit mass,
$\bar{f}$ = volume (body) forces per unit mass

and since mass is conserved along the motion, defining $\bar{f}_v \doteq \rho\bar{f}$ volume forces per unit volume

$$\rho\frac{D\bar{v}}{Dt} = \mathrm{div}\,\underline{\underline{\sigma}} + \bar{f}_v$$

The equation is cast in terms of quantities per unit volume as this is customary for solid elasticity, where density changes are first-order infinitesimals, while in fluid dynamics the form per unit mass is preferred.

Note that the second Euler's equation (angular momentum balance eq.) always holds once the first one is satisfied, thanks to the symmetry of $\underline{\underline{\sigma}}$ (no spin).

Now if we consider displacements $\bar{u}$ to be first-order infinitesimal quantities, and the same for the derivatives of $\bar{u}$, then

$$\bar{v} = \frac{\partial \bar{u}}{\partial t} \quad \text{and} \quad \bar{a} = \frac{\partial \bar{v}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial t^2}$$

since the $(\bar{v}\cdot\mathrm{grad})$ term yields a higher-order infinitesimal, hence in Euler's equation

$$\rho \frac{D\bar{v}}{Dt} \simeq (\rho_e + \rho')\frac{\partial^2 \bar{u}}{\partial t^2} \simeq \rho_e \frac{\partial^2 \bar{u}}{\partial t^2}$$

as again $\rho' \frac{\partial^2 \bar{u}}{\partial t^2}$ is higher-order

and thus finally

$$\rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = \mathrm{div}\,\underline{\underline{\sigma}} + \bar{f}_v \; ;$$

now we split both surface and volume forces as follows:

$$\bar{f}_v = \bar{f}_{v_0} + \bar{f}_{v_1}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_0 + \underline{\underline{\sigma}}_1$$

where $\bar{f}_{v_0}, \underline{\underline{\sigma}}_0$ are <u>finite</u> terms in the displacements $\bar{u}$,

$\bar{f}_{v_1}, \underline{\underline{\sigma}}_1$ are <u>infinitesimal</u> terms in the displacements $\bar{u}$.

For geophysical problems such as wave propagation through the Earth (which we shall address next), body forces are essentially gravity and Coriolis force, hence

$$\bar{f}_v = (\rho_e+\rho')\bar{g} + 2(\rho_e+\rho')\bar{v}\times\bar{\Omega} \simeq (\rho_e+\rho')\bar{g} + 2\rho_e\bar{v}\times\bar{\Omega}$$ (ignoring the higher-order term)

so that

$$\bar{f}_{v_0} = \rho_e \bar{g} \; ; \quad \bar{f}_{v_1} = \rho'\bar{g} + 2\rho_e \bar{v}\times\bar{\Omega}$$

where $\bar{g}$ = effective gravitational acceleration, including the rotating-Earth correction,

$\bar{\Omega}$ = angular velocity vector for Earth's rotation.

Summarizing,

$$\rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = \mathrm{div}\,\underline{\underline{\sigma}}_0 + \mathrm{div}\,\underline{\underline{\sigma}}_1 + \bar{f}_{v_0} + \bar{f}_{v_1}$$

and separating finite and infinitesimal terms to impose equality at all orders,

$$\mathrm{div}\,\underline{\underline{\sigma}}_0 + \bar{f}_{v_0} = \phi \qquad (1)$$

$$\rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = \mathrm{div}\,\underline{\underline{\sigma}}_1 + \bar{f}_{v_1} \qquad (2)$$

(1) describes ELASTOSTATICS: equilibrium of an elastic body under the effect of external forces,

(2) describes ELASTODYNAMICS: motion of the body for sufficiently small displacements.

Let us work out further the elastodynamics equation. We shall omit subscripts e, 1, and we shall also assume $f_{v,i} \approx \phi$, i.e. body force contribution to be negligible (which we prove later). The equation still contains different unknown quantities i.e. u_i, $\underline{\underline{\sigma}}$, and we want to reduce it to an eq. for the displacements u_i, using the constitutive law; since

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij} = \partial_j \left(2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk}\delta_{ij}\right) =$$

$$= 2\mu \partial_j \varepsilon_{ij} + \lambda \partial_i \varepsilon_{kk} = \underbrace{\mu \partial_j \partial_j u_i}_{\nabla^2 u_i} + \underbrace{\mu \partial_j \partial_i u_j}_{=\partial_i \partial_j u_j = \partial_i(\operatorname{div}\bar{u})} + \frac{1}{2}\lambda \partial_i \overbrace{\left(\partial_k u_k + \partial_k u_k\right)}^{2\partial_i(\operatorname{div}\bar{u})} =$$

$$= (\mu+\lambda)\,\partial_i \operatorname{div}\bar{u} + \mu \nabla^2 u_i$$

$$\Rightarrow \boxed{\rho \frac{\partial^2 u_i}{\partial t^2} = (\mu+\lambda)\,\partial_i \operatorname{div}\bar{u} + \mu \nabla^2 u_i}$$

Navier equation

$$\text{or} \quad \boxed{\rho \frac{\partial^2 \bar{u}}{\partial t^2} = (\mu+\lambda)\operatorname{grad}(\operatorname{div}\bar{u}) + \mu \nabla^2 \bar{u}}$$

Notes:
- We have assumed λ, μ to be uniform (position-independent), and remember that $\rho = \rho_e$, $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_1$ i.e. infinitesimal in $\bar{u}$.
- Since only ρ_e enter the equation the Navier eq. is effectively independent on the continuity equation containing ρ' (not the case for fluids!).
- The Navier eq. is a partial differential eq, therefore the solution of a specific problem requires stating proper boundary conditions, i.e. specifications about displacements or stresses at the boundaries of the domain.