Euler's equation (a sneak peek on fluid dynamics)

Euler's equation describes the dynamics of an ideal fluid (i.e. a fluid exempt from shear stresses and heat transport phenomena). We can derive it using what we previously learnt about integral and local forms of physical laws; the first law of motion for a fluid element is

$$\frac{D}{Dt}\int_{R(t)} \rho \bar{v}\, d^3x = \bar{F}(t) = \int_{R(t)} \rho \bar{f}\, d^3x$$

where $\bar{F}$ is the resultant of all forces acting on the region $R(t)$ of fluid and $\bar{f}$ is its associated force per unit mass

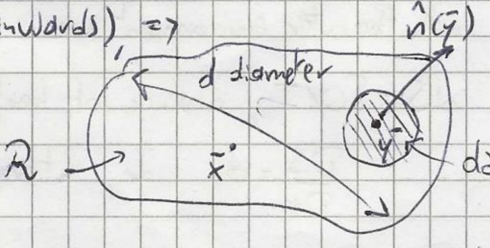
and since ρ obeys the continuity equation,

$$\frac{D}{Dt}\int_{R(t)} \rho \bar{v}\, d^3x = \int_{R(t)} \rho \frac{D\bar{v}}{Dt}\, d^3x = \int_{R(t)} \rho \bar{f}\, d^3x \Rightarrow \frac{D\bar{v}}{Dt} = \bar{f}$$

as $R(t)$ is arbitrary

So we have to make $\bar{f}$ more explicit. As seen before, what really matters is <u>surface</u> (or <u>contact</u>) forces, applied on the boundary surface ∂R of R; since we deal with an ideal fluid, it is just about pressure (normal force pointed inwards) $\Rightarrow$

$$\bar{F} = \int_{\partial R} -p(\bar{y})\, \hat{n}(\bar{y})\, da = \int_{\partial R} -p(\bar{y})\, d\bar{a}$$



where the integral spans the whole surface, i.e. $\forall \bar{y} \in \partial R$ where there will be a certain pressure $p(\bar{y})$ with orientation opposing the local outwardly-oriented normal vector $\hat{n}$ of the surface element da. In the limit where the diameter d of R becomes infinitesimal (around the position $\bar{x}$) we obtain $\bar{F}(\bar{x})$:

$$\bar{F}(\bar{x}) = \lim_{d\to 0}\left[-\int_{\partial R} p(\bar{y})\, d\bar{a}\right] = \lim_{d\to 0}\left[-\int_{R} \operatorname{grad} p\, d^3x\right] = -\operatorname{grad} p(\bar{x})\, V(\bar{x})$$

with $V(R)$ the volume of R

using a special case of the divergence theorem (see Appendix)

$$\int_{\partial V} f(\bar{x})\, d\bar{a} = \int_{V} \operatorname{grad} f(\bar{x})\, dV$$

for infinitesimal volume, the integral is approximated as the integrand times the volume (integrand $\simeq$ uniform)

$$\Rightarrow \bar{F}(\bar{x}) = \bar{f}(\bar{x})\, M(R) = -\operatorname{grad} p(\bar{x})\, V(R)$$

with $M(R)$ mass of the fluid element contained within the region R

$$\Rightarrow \bar{f}(\bar{x}) = -\mathrm{grad}\, p \, \frac{V(R)}{M(R)} = -\frac{1}{\rho(\bar{x})}\, \mathrm{grad}\, p(\bar{x}) \quad \text{so that we finally get}$$

$$\lim_{d\to\phi} \frac{V(R)}{M(R)} = \frac{1}{\rho(\bar{x})} = \sigma(\bar{x})$$

$$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\,\mathrm{grad}\, p \quad \text{or even more explicitly}$$

$$\boxed{\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\mathrm{grad})\,\bar{v} = -\frac{1}{\rho}\,\mathrm{grad}\, p} \quad \text{Euler's equation}$$

If the fluid is immersed in an external force field, and the most obvious example is a gravitational field (e.g. the Earth's one), we must add $M(R)\bar{g}$ to $\bar{F}$ (this will be a volume force) and the total force per unit mass will be $\bar{f}+\bar{g}$, so that Euler's eq. is modified as follows:

$$\boxed{\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\mathrm{grad})\,\bar{v} = -\frac{1}{\rho}\,\mathrm{grad}\, p + \bar{g}} \quad \text{Euler's eq. in a gravitational field}$$

and this form is valid for any volume force field acting on the fluid.

The fact that in an ideal fluid there is no heat exchange among parts of the fluid or with the outer environment means that the motion is ADIABATIC throughout the fluid. Since $dS = \delta Q/T$, absence of heat exchange means

$$\frac{DS}{Dt} = \phi \quad \text{or, in terms of } s \text{ entropy per unit mass, } \boxed{\frac{Ds}{Dt} = \phi}\,;$$

the entropy of any fluid particle is constant along the particle motion. This is the condition for adiabatic motion. Exploiting the known relationship between properties per unit mass "g" and per unit volume "ρg", $\rho \frac{Dg}{Dt} = \frac{\partial}{\partial t}(\rho g) + \mathrm{div}(\rho g \bar{v})$,

$$\frac{Ds}{Dt} = \phi \iff \boxed{\frac{\partial}{\partial t}(\rho s) + \mathrm{div}(\rho s \bar{v}) = \phi}$$

that is a "continuity equation" for entropy, or precisely for ρs (entropy volumetric density), where $\rho s \bar{v}$ is hence the entropy flux density.

It is not uncommon to find that entropy is uniform throughout the whole fluid at a certain time instant. Then the adiabatic condition, $Ds/Dt = \phi$, is simpler as it just becomes $\boxed{s = \text{constant}}$ ISENTROPIC FLOW CONDITION

Then if we consider w enthalpy per unit mass, thermodynamics tells us that

$$dw = T\,ds + \sigma\,dp = \text{for isentropic flow} = \sigma\,dp = \frac{1}{\rho}\,dp$$

and by the very definition of the gradient operator

$$\text{grad}\, w = \frac{1}{\rho}\,\text{grad}\, p$$

$$\Rightarrow \boxed{\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\text{grad}\,\bar{v}) = -\,\text{grad}\, w}$$ Euler's eq. for isentropic flow

Notice that an alternative form of this equation can be obtained. If $\bar{a}, \bar{b}$ are vectors,

$\text{grad}\,(\bar{a}\cdot\bar{b}) = (\bar{a}\cdot\text{grad})\bar{b} + (\bar{b}\cdot\text{grad})\bar{a} + \bar{a}\times\text{curl}\,\bar{b} + \bar{b}\times\text{curl}\,\bar{a}$; with $\bar{a} = \bar{b} = \bar{v}$,

$\text{grad}(v^2) = 2(\bar{v}\cdot\text{grad}\,\bar{v}) + 2\,\bar{v}\times\text{curl}\,\bar{v} \Rightarrow$

$$\text{grad}\left(\frac{1}{2}v^2\right) = \underline{(\bar{v}\cdot\text{grad})\bar{v}} + \bar{v}\times\text{curl}\,\bar{v}$$

and let us use this in Euler's eq. $\Rightarrow$

$$\Rightarrow \frac{\partial \bar{v}}{\partial t} + \text{grad}\left(\frac{1}{2}v^2\right) - \bar{v}\times\text{curl}\,\bar{v} = -\text{grad}\, w \Rightarrow \frac{\partial \bar{v}}{\partial t} - \bar{v}\times\text{curl}\,\bar{v} = -\text{grad}\left(\frac{1}{2}v^2 + w\right)$$

Now let us take the curl of both sides, since $\text{curl}(\text{grad}) = \phi$ identically,

$$\boxed{\frac{\partial}{\partial t}(\text{curl}\,\bar{v}) = \text{curl}\,(\bar{v}\times\text{curl}\,\bar{v})}$$

Take note that $\bar{\omega} = \text{curl}\,\bar{v}$ is called VORTICITY (a measure of rotational motion).

Finally, notice that in order to find a complete solution to the (thermo)dynamical problem of an ideal fluid in motion, we need complete knowledge over $(\bar{v}, \rho, p)$, i.e., all in all, 5 scalar quantities, and up to here we have exactly supplied **5** scalar equations:

Euler's equation, (3)
+
continuity equation, (1)
+
adiabatic equation (1)

Also any specific problem is well defined only when its boundary conditions (and initial conditions) are supplied. Boundary conditions (b.c.) should be discussed in particular: Since we are dealing with an ideal fluid and thus no friction can occur, the fluid must be allowed to flow freely along any solid boundary surface, while it cannot penetrate solid objects; in other words, we must set <u>zero</u> normal velocity component relative to the boundary, while no condition is imposed on the parallel velocity component (<u>FREE-SLIP CONDITION</u>):

$(\bar{v}_{fluid} - \bar{v}_{body})\cdot\hat{n}_{body} = \phi$ at any solid body surface (wall or obstacle in the flow).

Appendix: Divergence theorem - special cases

When some suitable conditions are met (V is a subset of $\mathbb{R}^3$, compact and with a piecewise smooth boundary $S \equiv \partial V$; $\bar{F}$ is a continuously differentiable vector field defined on V), then

$$\int_{S \equiv \partial V} \bar{F} \cdot d\bar{a} = \int_V \operatorname{div} \bar{F} \, dV \qquad \text{divergence (or Gauss's) theorem}$$

A case of special interest to us is $\bar{F}(\bar{x}) = f(\bar{x})\,\bar{c}$ i.e. the product of a scalar function $f(\bar{x})$ and a <u>constant vector</u> $\bar{c} \neq \bar{\phi}$. Then

$$\int_V \operatorname{div} \bar{F} \, dV = \int_V \operatorname{div}(f(\bar{x})\bar{c})\, dV = \int_V \Big[f(\bar{x}) \underbrace{\operatorname{div} \bar{c}}_{\phi} + \bar{c} \cdot \operatorname{grad} f(\bar{x}) \Big] dV = \bar{c} \cdot \int_V \operatorname{grad} f(\bar{x})\, dV$$

while $\displaystyle\int_{\partial V} \bar{F} \cdot d\bar{a} = \int_{\partial V} f(\bar{x})\, \bar{c} \cdot d\bar{a} = \bar{c} \cdot \int_{\partial V} f(\bar{x})\, d\bar{a}$ and equating these two we get

$$\bar{c} \cdot \left[\int_{\partial V} f(\bar{x})\, d\bar{a} - \int_V \operatorname{grad} f(\bar{x})\, dV \right] = \phi \quad \text{valid } \forall\, \bar{c} \neq \bar{\phi} \Rightarrow$$

$$\boxed{\int_{\partial V} f(\bar{x})\, d\bar{a} = \int_V \operatorname{grad} f(\bar{x})\, dV}$$

We can prove yet another case: If $\bar{F}(\bar{x}) = \bar{c} \times \bar{P}(\bar{x})$ with $\bar{c} \neq \bar{\phi}$ constant vector,

$$\int_V \operatorname{div} \bar{F}(\bar{x})\, dV = \int_V \operatorname{div}(\bar{c} \times \bar{P}(\bar{x}))\, dV = \int_V \Big[\bar{P}(\bar{x}) \cdot \underbrace{\operatorname{curl} \bar{c}}_{\phi} - \bar{c} \cdot \operatorname{curl} \bar{P}(\bar{x}) \Big] dV = -\,\bar{c} \cdot \int_V \operatorname{curl} \bar{P}(\bar{x})\, dV \,;$$

$$\int_{\partial V} \bar{F} \cdot d\bar{a} = \int_{\partial V} (\bar{c} \times \bar{P}(\bar{x})) \cdot d\bar{a} \underset{d\bar{a} = \hat{n}\, da}{=} \int_{\partial V} (\bar{P}(\bar{x}) \times \hat{n}) \cdot \bar{c}\, da = \bar{c} \cdot \int_{\partial V} \bar{P}(\bar{x}) \times \hat{n}\, da$$

and again since $\bar{c}$ is arbitrary,

$$\boxed{\int_V \operatorname{curl} \bar{P}(\bar{x})\, dV = -\int_{\partial V} \bar{P}(\bar{x}) \times d\bar{a}}$$