Back to statics

Static equilibrium requires balance of forces, hence for a fluid, setting $D\vec{v}/Dt = \phi$ means, in terms of Euler's equation,

$$\boxed{-\frac{1}{\rho}\,\mathrm{grad}\,p + \vec{g} = \phi}$$ mechanical equilibrium condition for an ideal fluid (also hydrostatic equilibrium condition)

Trivially, in the absence of external forces ($\vec{g} = \phi$) equilibrium implies $\mathrm{grad}\,p = \phi$, i.e. uniform pressure throughout the fluid.

Let us detail some consequences stemming from the equilibrium condition.

If there is a gravitational field $\vec{g}$, we can define a gravitational potential u such that the field is $\boxed{\vec{g} = -\mathrm{grad}\,u}$; in the simplest approximation for the Earth's field,

$u = gz \Rightarrow \vec{g} = -\mathrm{grad}\,u = -g\hat{e}_z$. Hence equilibrium states that

$$-\frac{1}{\rho}\,\mathrm{grad}\,p = \mathrm{grad}\,u$$

i.e. the gradients of p and u lie in the same direction, $\Rightarrow$

* Isobaric surfaces are also equipotential surfaces of the gravitational field.
* Applying the curl, $\mathrm{curl}(-\mathrm{grad}\,p) = \mathrm{curl}(\rho\,\mathrm{grad}\,u) \Rightarrow = \phi$

  $(\mathrm{curl}(\mathrm{grad}) = \phi)$ so $\mathrm{curl}(\rho\,\mathrm{grad}\,u) = (\mathrm{grad}\,\rho)\times(\mathrm{grad}\,u) + \rho\,\mathrm{curl}(\mathrm{grad}\,u) = \phi$

  but $(\mathrm{grad}\,\rho)\times(\mathrm{grad}\,u) = \phi$ means that $\mathrm{grad}\,\rho$ is also parallel to $\mathrm{grad}\,u$, $\mathrm{grad}\,p$

  $\Rightarrow$ surfaces of uniform density (isopycnic surfaces) coincide with isobaric and equipotential surfaces. Since all thermodynamic properties can be obtained from the pair (p, ρ), the same holds for their surfaces of constant value, and a fluid like this is called a BAROTROPIC FLUID.
* As Earth's gravity is vertical ($\vec{g} = -g\hat{e}_z$), equilibrium yields equipotential surfaces that are horizontal planes and all thermodynamic properties are a function of elevation $f(z)$; equilibrium reads $$-\frac{1}{\rho(z)}\frac{dp}{dz} = g$$
* As isobaric surfaces are equipotential, the interface between immiscible fluids (e.g. the free surface of a water basin) is an equipotential surface
* Notice that $\vec{g} = -g\hat{e}_z$ is an approximation; in particular, in a non-inertial reference frame (accelerated system), fictitious forces (the forces seen in the reference frame due to the frame being accelerated) must be included together with $\vec{g}$ in the equilibrium condition.

Let us see a few examples.

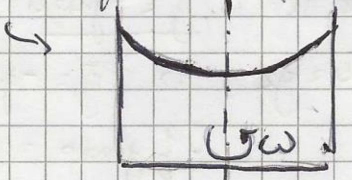
⊙ Centrifuge

Let us consider a centrifuge as a cylindrical tank rotating at constant angular velocity ω. In the centrifuge there is a fluid, let us say water. If the tank does not rotate, the separation surface water-air (free surface), that is an equipotential surface at equilibrium, is flat and horizontal. When the centrifuge rotates, in the rotating reference frame the potential is the sum of gravitational and centrifugal potential; indeed the force field per unit mass is

$\bar{g}' = \bar{g} + \omega^2 \bar{r} \Rightarrow$ as $\bar{g}' = -\text{grad}\, u'$, $u' = gz - \frac{1}{2}\omega^2 r^2$,

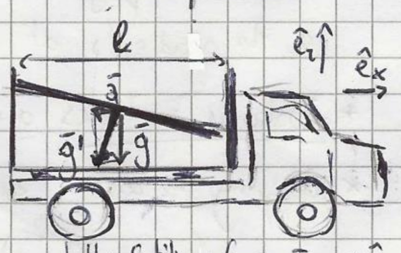
and isobaric surfaces (as the free surface is) are equipotential, i.e. satisfy $u' = \text{constant}$

$\Rightarrow gz - \frac{1}{2}\omega^2 r^2 = \text{constant} \Rightarrow z = \frac{1}{2g}\omega^2 r^2 + \text{constant}$ is the equation of a paraboloid with lowest point in the center.



⊙ Accelerating truck

Imagine a truck carrying a liquid in its load bed; let us say the bed is long $\ell = 4\,m$ and is filled to a height $h = 1.5\,m$. At the start and until it reaches a constant speed, its initial acceleration is $a = 1.5\,m/s$. Hence in its own reference frame the force $\bar{g}'$ takes into account the fictitious force $\bar{a} = -a\hat{e}_x$, $\Rightarrow$



$\bar{g}' = -g\hat{e}_z - a\hat{e}_x \Rightarrow u' = ax + gz$ and the free surface will be $u' = ax + gz = \text{constant}$, so of the type $z(x) = -\frac{a}{g}x + k$ (free surface $\perp \bar{g}'$). The constant is determined imposing constant volume: for parallelepipedal load bed, $h\ell = \frac{1}{2}\left[\left(-\frac{a}{g}L + k\right) + k\right]\ell \Rightarrow k = h + aL/2g$; $\Rightarrow z(0) = h + a\ell/2g = 1.8\,m$, $z(\ell) = h - a\ell/2g = 1.2\,m$.

⊙ Self-gravitating star (non-rotating, non-relativistic)

A huge mass like a star is a fluid very much held together by its own gravity (self-gravitating object). Density cannot be uniform due to the high pressure on the gas. The Newtonian gravitational potential u obeys Poisson's equation $\nabla^2 u = 4\pi G\rho$ (like ϕ electrostatic potential, a $\sim 1/r^2$ force field) where $\nabla^2 u = \text{div}(\text{grad}\, u)$. Hydrostatic equilibrium (not necessarily a thermal equilibrium) states

$-\frac{1}{\rho}\text{grad}\, p = \text{grad}\, u \Rightarrow$ applying the divergence, $\text{div}(\text{grad}\, u) = -\text{div}\left(\frac{1}{\rho}\text{grad}\, p\right) = 4\pi G\rho$.

In the absence of rotation, there is a spherical symmetry $\Rightarrow$

$$\text{div}\left(\frac{1}{\rho}\text{grad}\, p\right) = -4\pi G\rho \quad \text{becomes} \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho(r)$$

[For a solution, see the Appendix: Eddington solar model.]

Equilibrium of the atmosphere - Stability of the equilibrium

For large masses of fluid, such as a planetary atmosphere, the mechanical equilibrium condition $\text{grad}\,p = \rho\bar{g}$ cannot be integrated straight away because ρ is not uniform. In order to derive and explicit trend for $p(z)$ and hence for other thermodynamic properties, a range of (more or less acceptable) approximations and assumptions can be put into play.

* Mechanical + thermal equilibrium (isothermal atmosphere) (dry atmosphere)

We are assuming that atmosphere mixing brought T to be uniform.

=> the Gibbs free energy per unit mass $\phi = G/M$ obeys:

$d\phi = -s\,dT + v\,dp = v\,dp = \frac{1}{\rho}dp$; by definition of gradient,

$\frac{1}{\rho}dp = d\phi \iff \frac{1}{\rho}\text{grad}\,p = \text{grad}\,\phi \Rightarrow \text{grad}\,\phi = \bar{g} = -\text{grad}(gz)$

$\Rightarrow \text{grad}(\phi + gz) = \phi \Rightarrow \boxed{\phi + gz = \text{constant}}$ thermodynamical equilibrium condition for the atmosphere in an external (vertical) field: The sum of Gibbs free energy and gravitational potential energy is constant throughout the fluid.

Scale height of the Earth's atmosphere

As a consequence of an isothermal atmosphere (a somewhat acceptable assumption: Over the lowest 70 km of height, $T = \tilde{T} = 250\,K$ within a 15% maximum deviation), one can estimate the pressure trend with height.

Let us put together the equilibrium condition $\frac{dp}{dz} = -\rho g$ with the ideal gas law $pV = NR\tilde{T}$ and the assumption $T = \tilde{T}$ constant.

now $\rho = \frac{1}{v} = \frac{M}{V} \Rightarrow V = M/\rho$ and

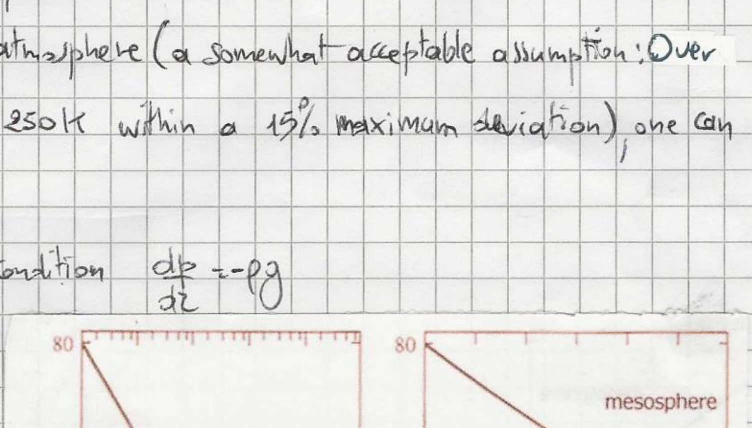
$\frac{pM}{\rho} = NR\tilde{T}$ or $\frac{p}{\rho} = R\tilde{T}/M_{mol}$

($M_{mol} \doteq M/N$ molar mass); inverting for ρ,

$\frac{dp}{dz} = -g\frac{M_{mol}}{R\tilde{T}}p$;

let us define $\boxed{H \doteq R\tilde{T}/gM_{mol}}$ SCALE HEIGHT

$\Rightarrow \frac{dp}{dz} = -\frac{1}{H}p \Rightarrow$



Mean p and T vs altitude (at equinox, temperate regions). Source: "Introduction to Atmospheric Chemistry" Daniel J. Jacob, Princeton University Press, 1999.

$\underline{p(z) = p_0 e^{-z/H}}$, with $p_0 = p(z=\phi)$ pressure at sea level.

So we see that pressure decreases exponentially with a decay length that is precisely the scale height H (in other terms, H gives us a scale of the atmosphere's thickness).

Plugging in the data $R = 8.31\ J/mol.K$,

$\tilde{T} = 250\ K$, $g = 9.81\ m/s^2$,

$M_{mol}(air) = 28.96 \cdot 10^{-3}\ Kg/mol$

$\Rightarrow H = 7.3\ Km$

which is quite approximate, but pretty much in the ball park (see D.J. Jacob's diagram).

* Isentropic (dry) atmosphere

Let us have a less stringent hypothesis than the isothermal one, and namely, an <u>isentropic</u> atmosphere. We shall justify this assumption on the basis of the following arguments.
First of all, the heat conduction coefficient of gases is very low (heat transport would mainly occur by convection), so we may assume that a virtual vertical displacement of an air parcel in a situation of hydrostatic equilibrium would be <u>adiabatic</u>, and ideally slow ⇒ reversible and thus isentropic. Second, we can also assume that there is continuous air mixing in the troposphere ⇒ upon vertical displacement of the air element, its entropy coincides with that of its surroundings. Hence we conclude that the atmosphere is <u>isentropic</u>.

Now if we consider the enthalpy per unit mass $w = H/M$,

$dw = \cancel{T ds} + v dp = \frac{1}{\rho} dp$ ⇒ by definition of gradient $\text{grad}\, w = \frac{1}{\rho} \text{grad}\, p$

⇒ the mechanical equilibrium condition becomes

$$\text{grad}\, w = \bar{g} = -\text{grad}(gz) \Rightarrow \text{grad}(w + gz) = \phi$$

$$\Rightarrow \boxed{w(z) + gz = \text{constant}}$$ with $w_0 = w(z=\phi)$ enthalpy at the ground

equilibrium = sum of enthalpy and gravitational potential energy is constant across the fluid.

In the ideal gas approximation $dw = c_p dT$ with c_p = specific heat per unit mass at constant p

$\Rightarrow w = c_p T$ (if c_p constant, otherwise $w = \int c_p(T) dT$)

$\Rightarrow c_p T(z) + gz = c_p T_0$ with $T_0 = T(z=\phi)$

$$\Rightarrow \boxed{T(z) = T_0 - \frac{g}{c_p} z}$$ linear decrease of temperature with height(*)

The trends $p(z)$, $\rho(z)$ can be obtained from $T(z)$ equation combined with the equation for an adiabatic process (which is coherent with the current hypothesis of isentropic atmosphere):

- $T p^{\frac{1}{\gamma}-1} = \text{constant}$; $1/\gamma - 1 = C_v/C_p - 1 = (C_v - C_p)/C_p = -R/C_p$ (<u>molar</u> specific heats C_v, C_p!)

$\Rightarrow T / p^{R/C_p} = \text{constant} = T_0 / p_0^{R/C_p}$

$$\Rightarrow \boxed{\frac{p(z)}{p_0} = \left(\frac{T(z)}{T_0}\right)^{C_p/R}}$$

(*) = Notice that this approximation is limited: T would become negative for $z > c_p T_0 / g$!
with $c_p = 1005$ J/kg·K at 1 atm for dry air, $z = c_p T_0 / g \simeq 30$ km

• $TV^{\gamma-1} = \text{constant}$; $\gamma - 1 = C_p/C_v - 1 = (C_p - C_v)/C_v = R/C_v$

$\Rightarrow T/\rho^{\gamma-1} = T/\rho^{R/C_v} = \text{constant} = T_0/\rho_0^{R/C_v}$

$$\Rightarrow \boxed{\frac{\rho(z)}{\rho_0} = \left(\frac{T(z)}{T_0}\right)^{C_v/R}}$$

The mechanical equilibrium condition for a dry isentropic atmosphere is often written in a slightly different way. Let us take the d/dz derivative of the condition

$$T(z) = T_0 - \frac{g}{C_p} z \longrightarrow \boxed{\frac{dT}{dz} = -\frac{g}{C_p}} \simeq 10^{-2}\ K \cdot m^{-1}$$

($C_p = 1005\ J/kg \cdot K$ at 1 atm for dry air)

Also notice that:

⊙ This is an equilibrium value for the temperature drop with height; we should ask ourselves when this equilibrium is <u>stable</u>.

⊙ The air is loaded with water vapor – in varying amounts for different parts of the planet – and if a parcel of humid air rises and gets colder, water condenses and releases its latent heat of vaporization. Also keep in mind that $C_p \simeq 2050\ J/kg \cdot K$ for humid air (~ twice as much than dry air) ⇒ as a consequence the temperature vertical gradient in temperate regions take lower values ($\sim 5 \cdot 10^{-3}\ K \cdot m^{-1}$) than the estimate above (rather valid for very dry areas... like Antarctica).

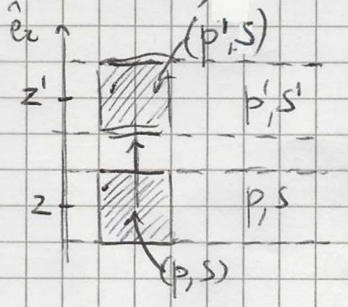
Stability

As mechanical equilibrium does not imply thermal equilibrium, the stability question arises. If the equilibrium is not stable, currents can occur (CONVECTION) that mix the fluid in order to make T uniform. In other words, the stability condition for mechanical equilibrium is the absence of convection, which we shall derive in the following.

Let us take a fluid element of specific volume υ at height z, and thus with thermodynamic properties $f(z)$; $\Rightarrow \upsilon = \upsilon(p(z), s(z))$.

An adiabatic vertical displacement of the fluid element will cause its specific volume to change from $\upsilon(p,s)$ to $\upsilon(p',s)$ (s = constant during an adiabatic process);

the element, now at a different height, will find itself in a region with values (p', s')



(= see figure). A necessary condition for equilibrium stability is that if the element rises, its density ρ_{el} must be higher than its surrounding environment (and hence it falls back), and vice versa if it moves down it should be less dense than the environment (and hence it goes back up). For a displacement dz,

$dz > \phi \quad \rho_{ext}(p', s') \leq \rho_{el}(p', s) \quad$ or $\quad v_{ext}(p', s') \geq v_{el}(p', s)$

$dz < \phi \quad \rho_{ext}(p', s') \geq \rho_{el}(p', s) \quad$ or $\quad v_{ext}(p', s') \leq v_{el}(p', s)$

i.e. $dv/dz \geq \phi$ for the atmosphere (environment); for infinitesimal displacement dz,

$$dv = \left(\frac{\partial v}{\partial s}\right)_{p=\text{const}} \cdot \frac{ds}{dz} dz \Rightarrow \text{stability requires} \quad \boxed{\frac{dv}{dz} = \left(\frac{\partial v}{\partial s}\right)_p \frac{ds}{dz} \geq \phi}$$

Now let us write $\left(\frac{\partial v}{\partial s}\right)_p = \left(\frac{\partial v}{\partial T}\right)_p \left(\frac{\partial T}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial v}{\partial T}\right)_p \Rightarrow$

$$\frac{T}{c_p}\left(\frac{\partial v}{\partial T}\right)_p \frac{ds}{dz} \geq \phi$$

; T/c_p is positive, and typically matter expands when heated ($\Rightarrow (\partial v/\partial T)_p > \phi$)

$\Rightarrow$ the stability condition is $\boxed{\frac{ds}{dz} \geq \phi}$; hence let us elaborate further

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz} = \frac{c_p}{T}\frac{dT}{dz} - \left(\frac{\partial v}{\partial T}\right)_p \frac{dp}{dz} \geq \phi$$

↳ 4th Maxwell relation

Mechanical equilibrium requires $\frac{dp}{dz} = -\rho g = -\frac{1}{v} g$

$\Rightarrow \frac{c_p}{T}\frac{dT}{dz} + \left(\frac{\partial v}{\partial T}\right)_p g/v \geq \phi$; since $\beta \equiv \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p$ coefficient of thermal expansion

$$\Rightarrow \boxed{-\frac{dT}{dz} \leq \frac{g\beta T}{c_p}}$$

CONDITION FOR ABSENCE OF CONVECTION = STABLE EQUILIBRIUM CONDITION

i.e. the decrease rate of T with height must not exceed $g\beta T/c_p$. Notice that

* for an ideal gas $\beta = 1/T \Rightarrow -dT/dz \leq g/c_p$;
* The equality condition means $ds/dz = \phi \Rightarrow s$ is uniform, i.e. the isentropic atmosphere case seen just before (where, for $\beta = 1/T$, $-dT/dz = g/c_p$).

APPENDIX: Eddington solar model (1926)

This model describes the Sun (or in general a star) using a number of simple (sometimes crude) assumptions, yet getting fairly close in the determination of some of the quantities of interest.

If we consider the star as a sphere of ionized gas, which self-gravitates due to the huge amount of matter, hydrostatic equilibrium (non-rotating star)

$$-\frac{1}{\rho}\,\mathrm{grad}\,p = \mathrm{grad}\,u \quad (\leftarrow \text{barotropic fluid})$$

where u is the Newtonian gravitational potential, which obeys a Poisson eq. ($\sim 1/r^2$)

$$\nabla^2 u = 4\pi G\rho$$

and since $\nabla^2 u = \mathrm{div}(\mathrm{grad}\,u)$ we apply the divergence to the hydrostatic equilibrium equation, reduced to a spherically-symmetric equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho$$

A relationship between p and ρ is needed. This may be complicated as it is related to energy transport in the star, which requires knowledge of fusion reactions generating energy and the transport mechanisms, i.e. convection and radiation, inside the star (which depend on complicated matters like to transparency/opacity of the star to radiation) up to its surface. Here comes Eddington's hypothesis: Internal energy transport is dominated by the radiation mechanism.

Then, we must consider that pressure is made of two contributions:

$$p = p_g + p_r$$

with p_g the gas pressure and p_r the radiation pressure. For p_g we have the ideal gas law

$$p_g V = NRT = N N_A k_B T = \frac{M}{\langle m\rangle} k_B T$$

(# moles · # particles/mole = # particles = total mass/avg mass)

and we call

$$\langle m\rangle = \mu m_p \quad (\mu = \text{relative molecular mass, } \langle m\rangle/m_p,\ m_p \text{ proton mass})$$

So

$$p_g = \rho k_B T/\mu m_p$$

For the radiation pressure, upon local thermodynamic equilibrium of radiation with the star's matter,

$$p_r = \frac{1}{3} \alpha T^4$$

and the two contributions are weighed with the parameter β:

$$\begin{cases} p_g = (1-\beta) p \\ p_r = \beta p \end{cases}$$

(Note: β is assumed to uniform... but it turns out to be quite ok.)

Let us put together the eqs. for p_g, p_r and the β - $1-\beta$ ratios:

$$p_r = \frac{1}{3}\alpha T^4 \rightarrow \quad T^4 = \frac{3}{\alpha} p_r = \frac{3}{\alpha}\beta p \longrightarrow \textcircled{=}$$

$$p_g = \rho K_B T / \mu m_p \rightarrow \quad T = \frac{\mu m_p}{\rho K_B} p_g \rightarrow \quad T^4 = \left(\frac{\mu m_p}{\rho K_B}\right)^4 p_g^4 = \left(\frac{\mu m_p}{\rho K_B}\right)^4 (1-\beta)^4 p^4$$

$$\Rightarrow p^3 = \left(\frac{K_B}{\mu m_p}\right)^4 \frac{3}{\alpha} \frac{\beta}{(1-\beta)^4} \rho^4 \Rightarrow \quad \boxed{p = K \rho^{4/3}} \quad \text{with } K \doteq \left[\frac{3}{\alpha}\left(\frac{K_B}{\mu m_p}\right)^4 \frac{\beta}{(1-\beta)^4}\right]^{\frac{1}{3}}$$

This equation of state has the form of a <u>polytrope</u>, a power law p-ρ generally expressed as

$$p = K\rho^\gamma = K\rho^{1+\frac{1}{n}} \quad (\text{hence } \rho \propto T^n)$$

Note that the adiabatic process is a special polytrope, where γ is the ratio of specific (constant p / constant V) heats and for a fully ionized (monoatomic) hydrogen plasma we would expect $\gamma = 5/3$ ($n = 1.5$) while here $\gamma = 4/3$ ($n=3$) which accounts for the radiation process and pressure. In general a 'polytropic star' is a star where the barotropic equation of state p-ρ is in the form $p = K\rho^{1+\frac{1}{n}}$.

We can reduce the problem to a dimensionless equivalent using the values $T_c = T(r=0)$, $p_c = p(r=0)$, $\rho_c = \rho(r=0)$ and defining

$\vartheta \doteq T/T_c$ dimensionless temperature.

Hence $\quad p_g = p(1-\beta) = K\rho^{4/3}(1-\beta) = \rho K_B T/\mu m_p \Rightarrow$

$$p^{1/3} = k_B T / \mu m_p K(1-\beta) \Rightarrow \frac{\rho}{\rho_c} = \left(\frac{T}{T_c}\right)^3 = \vartheta^3 \; ; \; \frac{p}{p_c} = \left(\frac{\rho}{\rho_c}\right)^{4/3} = \vartheta^4$$

Notice that $\vartheta(\phi) = 1$; $\vartheta(R) = \phi$

where we call $R =$ star radius, and $T, p, \rho = \phi$ at $r = R$ where there is no longer any matter.

A further boundary condition: since in $r = \phi$ mass is null and so is gravitational force

$$\left.\frac{dp}{dr}\right|_{r=\phi} = \phi \Rightarrow \frac{dp}{dr} \propto \frac{d\vartheta^4}{dr} \propto \vartheta^3 \frac{d\vartheta}{dr} = \phi \Leftrightarrow \boxed{\left.\frac{d\vartheta}{dr}\right|_{r=\phi} = \phi} \quad (\text{since } \vartheta(\phi) = 1).$$

With the dimensionless radial coordinate

$\xi \doteq r/a$ (with a a scale length suitably defined later)

the hydrostatic equilibrium equation can be reworked:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho \quad \leftarrow \quad r = a\xi$$

$$\Rightarrow \frac{1}{a^2\xi^2}\frac{1}{a}\frac{d}{d\xi}\left(\frac{a^2\xi^2}{\rho}\frac{1}{a}\frac{dp}{d\xi}\right) = -4\pi G\rho$$

$$\frac{1}{a^2\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{\rho}\frac{dp}{d\xi}\right) = -4\pi G\rho \quad \leftarrow \quad p/p_c = \vartheta^4 ; \; \rho/\rho_c = \vartheta^3$$

$$\Rightarrow \frac{1}{a^2\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{\rho_c\vartheta^3}p_c\frac{d\vartheta^4}{d\xi}\right) = \frac{p_c}{\rho_c}\frac{1}{a^2\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{\vartheta^3}4\vartheta^3\frac{d\vartheta}{d\xi}\right) = -4\pi G\rho_c\vartheta^3$$

$$\Rightarrow \frac{p_c}{\pi G\rho_c^2}\frac{1}{a^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\vartheta}{d\xi}\right) = -\vartheta^3$$

$p_c = K\rho_c^{4/3}$

$$\frac{K}{\pi G\rho_c^{2/3}}\frac{1}{a^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\vartheta}{d\xi}\right) = -\vartheta^3$$

and we can choose $a \doteq \left[K/\pi G\rho_c^{2/3}\right]^{1/2}$ so that the factor in front becomes 1.

The final form of the equation is

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\vartheta}{d\xi}\right) = -\vartheta^3} \quad \text{Lane-Emden equation}$$

or better, the Lane-Emden equation of polytropic index $n=3$. If we had a generic polytropic eq. of state $p = K\rho^{1+\frac{1}{n}}$ we would find the same form with n instead of 3. This eq. must be integrated to find $\vartheta(\xi)$, i.e. $T(r)$ and so $p(r)$, $\rho(r)$.

Unfortunately, analytical solutions to the LE eq. exist only for $n = 0, 1, 5$. It is worth spending a few words on these and other notable cases (including our case, $n=3$).

* $n=0$ — Straightforward integration yields (together with the aforementioned b.c.)

$$\vartheta(\xi) = 1 - \xi^2/6 \qquad \text{and} \qquad \xi_1 = \sqrt{6} \simeq 2.449$$

Physically, $n=0$ means $\rho = \rho_c$, INCOMPRESSIBLE star (and $p = p_c\vartheta$). That is hardly a model for a star, but it can account for a terrestrial planet (a planet, like the Earth that is mostly made out of rocky and metallic material, as opposed to gaseous planets like Jupiter and Saturn).

* $n=1$ — A smart mathematician will find

$$\vartheta(\xi) = \sin\xi/\xi \qquad \text{and} \qquad \xi_1 = \pi$$

Curiously, integrating the mass eq. $\frac{dm}{dr} = 4\pi r^2\rho$ in this case yields (with some algebra) $R = (\pi K/2G)^{1/2}$, i.e. the star radius does not depend on its mass M_*. As a note, it turns out that polytropes with $0.5 < n < 1$ can be used to model neutron stars.

* $n=5$ — A very smart mathematician will find

$$\vartheta(\xi) = 1/(1+\xi^2/3)^{1/2} \qquad \text{and} \qquad \xi_1 \to \infty \text{ (yet total mass is still finite)}$$

* $n>5$ - $n=\infty$ — $\forall\, n \geq 5$ the star has infinite radius. For $n \to \infty$, $p = K\rho$, which is the isothermal case. At large radii $\rho \sim 1/r^2$ and both radius and mass of the star are infinite. Physically, in order to have an isothermal sphere of finite radius, the sphere must be embedded in an external medium supplying a pressure such that an equilibrium can be reached. This can model the gas core of big structures such as molecular clouds.

* $n=1.5$ — No analytic solution but notice that it says $p = K\rho^{5/3}$, i.e. here we have a fully convective star, where heat transport is basically accomplished by a fast motion and mixing of fluid elements, in an adiabatic fashion (thus the adiabatic exponent $\gamma = 5/3$ for monoatomic gases). That can be considered true for low-mass stars, $M_* < 0.3 M_\odot$. Note: $RM_*^{1/3} \propto K/G$, i.e. a smaller star is more massive! $\xi = 3.654$. (e.g. nonrelativistic white dwarfs)

* $n=3$ — Finally, we get back to the Eddington model. Again no exact solution. $\xi_1 = 6.897$. Let us also consider the mass equation

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

(by numerical integration of LE → ϑ → ...)

and define $y(\xi) \doteq -\xi^2 d\vartheta/d\xi$ (that is a monotonically increasing function of the radial coordinate ξ, with $y_1 \doteq y(\xi_1) \doteq 2.018$). In dimensionless units

$$\frac{1}{a}\frac{dm}{d\xi} = 4\pi a^2 \xi^2 \rho_c \vartheta^3 = 4\pi a^2 \xi^2 \rho_c \frac{1}{\xi^2}\frac{dy}{d\xi}$$

$$\Rightarrow \quad \frac{dm}{d\xi} = 4\pi a^3 \rho_c \frac{dy}{d\xi} \quad \text{and} \quad m(\xi) = 4\pi a^3 \rho_c y \quad \text{(constant of integration} = \emptyset \text{ since } m(\emptyset) = \emptyset)$$

⇒ finally the star mass is $M_* = m(\xi_1) = m(R) = 4\pi \rho_c a^3 y_1$

and using the definitions of $a \doteq (K/\pi G \rho_c^{2/3})^{1/2}$ and $K \doteq \left[\frac{3}{a}\left(\frac{k_B}{\mu m_p}\right)^4 \frac{\beta}{(1-\beta)^4}\right]^{1/3}$

$$M_* = 4\pi (K/\pi G)^{3/2} y_1 = 4\pi \left[\frac{1}{(\pi G)^3}\frac{3}{a}\frac{\beta}{(1-\beta)^4}\right]^{1/2} \frac{1}{\mu^2} y_1$$

now we define a reference mass value

$$M_0 \doteq \left[\frac{1}{(\pi G)^3}\left(\frac{k_B}{m_p}\right)^4 \frac{3}{a}\right]^{1/2} 4\pi y_1 = 3.586 \cdot 10^{31}\ \text{Kg}$$

and $\zeta \doteq \mu^2 M_*/M_0$

so that $\zeta^2 = \beta/(1-\beta)^4$

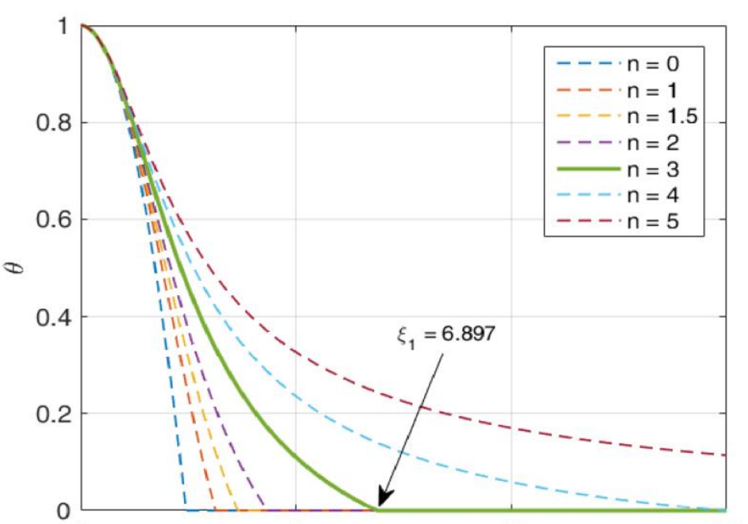
These results bring us to a few considerations.

⊙ $M_* = 4\pi (K/\pi G)^{3/2} y_1 = \left(\frac{K}{0.362\,G}\right)^{3/2}$ indicates that the star mass does not depend on the radius of the star itself (compare to $n=1$ where instead K did set a radius R, but not the mass M_*, and to a general case where we get a relation between R and M_* through K)!

⊙ The ratio between radiation and gas pressure, $\beta/(1-\beta)$, increases strongly with ζ and thus with both the star mass M_* and the mean molecular weight μ. With the observed solar $M_\odot = 1.989 \cdot 10^{30}$ Kg and $\mu = 0.68$ (the latter being obtained in the context of the more advanced Standard Solar Model (SSM), => $\zeta = 0.0256 \ll 1$ and inverting the ζ-β relation, $\beta \simeq \zeta^2 - \frac{1}{4}\zeta^4 + O(\zeta^6) = 6.577 \cdot 10^{-4}$. This says that p_r is a small contribution to $p = p_g + p_r$, and that radiative transport of energy is by far less effective than convection, where the latter occurs. But it turns out that in a star like the Sun, convection takes place only in the outermost region, accounting for a minor fraction of its bulk.

⊙ Despite its simple assumptions, Eddington's model yields a fairly accurate description of the thermodynamic quantities. Compared to the SSM, it gets values correct within a factor two:

Eddington (with observed values $M_\odot$ and $R = 6.960 \cdot 10^8$ m)		SSM
$T_c = \frac{\xi_1}{4 y_1} \mu (1-\beta) \frac{G M_\odot m_p}{R k_B}$	$= 1.34 \cdot 10^7$ K	$1.58 \cdot 10^7$ K
$\rho_c = \frac{\xi_1^3}{4\pi y_1} \frac{M_\odot}{R^3}$	$= 7.63 \cdot 10^4$ Kg/m^3	$15.6 \cdot 10^4$ Kg/m^3
$p_c = \frac{\xi_1^4}{16\pi y_1^2} \frac{G M_\odot^2}{R^4}$	$= 1.24 \cdot 10^{16}$ N/m^2	$2.38 \cdot 10^{16}$ N/m^2



Further info: ⊙ R. Fitzpatrick, 'Fluid Mechanics', available online at https://farside.ph.texas.edu, under 'Teaching'

⊙ C.J. Clarke and R.F. Carswell, 'Principles of Astrophysical Fluid Dynamics', Cambridge University Press