Dynamics of ideal fluids

Let us recall the fact that an IDEAL FLUID is a fluid such that all phenomena are REVERSIBLE. This excludes

- shear stresses (source of friction → irreversibility) ⇒ pressure is isotropic and $\sigma_{ij} = -p\,\delta_{ij}$;
- heat conduction (as heat transfer → irreversibility) ⇒ the motion of fluid elements is adiabatic and reversible, i.e. $Ds/Dt = \phi$ (or even $s = \text{constant}$ for isentropic flow).

Euler's equation, continuity equation and adiabatic equation supply us with 5 scalar conditions necessary to describe the flow with 5 scalar quantities $(\bar{v}, \rho, p)$.

To represent (and if we want, to visualize) the flow we can make use of concepts such as STREAMLINES, PATHLINES, STREAKLINES.

STREAMLINE is a line in the fluid at a certain time instant t that is tangent to the velocity vector at said time t in each of its points. In these terms, it is an Eulerian concept: when taking a "photograph" of the flow at a time t, we take a picture of the velocity field and visualize the streamlines.

Moving by a displacement $d\bar{x}$ along $\bar{v}$ (i.e. along a streamline) we can say $\bar{v} \times d\bar{x} = \phi$, or, in components

$$\left.\begin{aligned} v_y\,dz - v_z\,dy &= \phi \\ v_z\,dx - v_x\,dz &= \phi \\ v_x\,dy - v_y\,dx &= \phi \end{aligned}\right\} \Rightarrow \boxed{\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}}$$

differential equation for a streamline

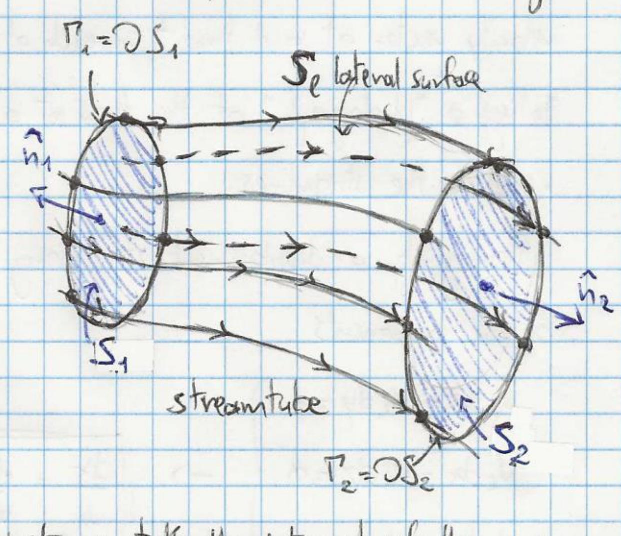
Notice that no intersections can occur for a streamline with itself or another one, otherwise the intersection point would have two different velocity vectors.

PATHLINE or TRAJECTORY is the curve traced out by the motion of a fluid particle as time progresses. In these terms, it is a Lagrangian concept (we follow the evolution of a certain fluid particle). Notice that a pathline can intersect itself (or another one, provided the intersecting points belong to different times), as a particle can visit the same place at different times. Notice also that <u>streamlines and pathlines coincide for STEADY or STATIONARY flow</u> ($\bar{v} = \bar{v}(\bar{x})$, i.e. $\partial\bar{v}/\partial t = \phi$). It is the solution to $d\bar{x}(t)/dt = \bar{v}(\bar{x}(t), t)$.

STREAKLINE is the locus of all fluid particles, at time t, that have passed in the past through a particular point $\bar{x}_0$. In other words, it is a solution to $d\bar{x}(t)/dt = \bar{v}(\bar{x}(t), t)$ with $\bar{x}(t_0) = \bar{x}_0$. In practice, it can be visualized by continuous injection of a dye into the fluid at position $\bar{x}_0$. Notice that a streakline cannot intersect itself or another one since two particles cannot occupy the same position at the same time instant, unless the origin of a streakline belongs to another streakline.

TIMELINE is the curve formed by a set of fluid particles that constituted a line at a certain time. It is the time evolution of a curve in the fluid, that is to say, the curve gets displaced as time goes by. Once again, it cannot intersect itself (points on the timeline are particles at the same time instant).

Let us take a closed curve Γ within the fluid, the set of streamlines passing through the points $\in \Gamma$ is called a STREAMTUBE. By definition of streamline, there is no flow through the wall of the streamtube.



The concept is most interesting for a steady flow, when streamlines = pathlines. Then, taking a closed curve Γ_1 and following the streamlines going through Γ_1 we can reach another closed curve Γ_2 along the flow, i.e. the streamtube. Steady state means that the continuity eq.

$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\bar{v}) = \phi$ becomes $\mathrm{div}(\rho\bar{v}) = \phi$. Let us take the integral of this over the streamtube volume portion between the two surfaces defined by the curves Γ_1, Γ_2,

$$\int_V \mathrm{div}(\rho\bar{v})dV = \int_{S=\partial V} \rho\bar{v}\cdot\hat{n}\,da = \phi \quad \text{with } S = S_1 \cup S_2 \cup S_e \text{ but flux through } S_e = \phi$$

$$\Rightarrow \int_{S_1} \rho\bar{v}\cdot\hat{n}_1\,da + \int_{S_2} \rho\bar{v}\cdot\hat{n}_2\,da = \phi$$

i.e. incoming flux through S_1 = outgoing flux through S_2. In other words, it is quite obvious that in a steady flow mass is conserved in a flux tube = streamtube. If we take S_1, S_2 small enough to approximate uniform $\rho, \bar{v}$ over them, and call v_1, v_2 the incoming/outgoing velocities, A_1, A_2 the respective areas of S_1, S_2, and ρ_1, ρ_2 the density values, then

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

We call $q = \dot{m} = \int_S \rho \bar{v} \cdot \hat{n} \, da$ MASS FLOW RATE $([q] = [M \cdot T^{-1}])$; so we can say that in a steady flow, the mass flow rate is conserved along a streamtube (coinciding with a flux tube). For incompressible flow, $\rho_1 = \rho_2$,

$$v_1 A_1 = v_2 A_2$$ (a fact already recognized by Leonardo da Vinci)

Momentum flux density tensor – Conservation of momentum

We recall the fact that $\bar{v}$ velocity is the momentum per unit mass of a fluid element, and $\rho\bar{v}$ is the momentum per unit volume. We want to analyze what happens to momentum in a region occupied by an ideal fluid in motion. We shall adopt two different starting points (obviously equivalent, in that they yield the same result). Following Landau's line of reasoning, and considering the i-th component ρv_i of the momentum per unit volume we can write

$$\frac{\partial}{\partial t}(\rho v_i) = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} \quad ; (*)$$

now let us use the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\bar{v}) = \phi \quad \sim \text{in tensor notation } \partial_t\rho + \partial_k(\rho v_k) = \phi$$

and Euler's equation

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\mathrm{grad})\bar{v} = -\frac{1}{\rho}\,\mathrm{grad}\,p - \mathrm{grad}\,u \quad \sim \text{in tensor notation } \partial_t v_i + v_k\partial_k v_i = -\frac{1}{\rho}\partial_i p - \partial_i u$$

$$\Rightarrow (*)\ \partial_t(\rho v_i) = -\rho v_k\partial_k v_i - \partial_i p - \rho\partial_i u - v_i\partial_k(\rho v_k) = \qquad \text{using } \partial_i p = \delta_{ik}\partial_k p$$

$$= -\rho v_k\partial_k v_i - v_i\partial_k(\rho v_k) - \partial_k p\,\delta_{ik} - \rho\partial_i u =$$

$$= -\partial_k(\rho v_i v_k - p\delta_{ik}) - \rho\partial_i u$$

We define $\Pi_{ik} \doteq \rho v_i v_k + p\delta_{ik}$ (notice that it is a symmetric tensor of order 2)

and summarize

$$\boxed{\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_k}\Pi_{ik} - \rho\frac{\partial u}{\partial x_i}}$$

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Let us take a step back (to finally get the same result). For an extensive quantity G with an associated quantity per unit mass $g \doteq G/M$ (and ρg per unit volume), we shall write the <u>flux</u> of G through a surface S, i.e. the amount of this property crossing S per unit time. If we take an infinitesimal surface dS with normal unit vector $\hat{n}$ and area da, a displacement $d\bar{x}$ in an infinitesimal time period dt yields

$$d\phi(G) = \underset{\frac{G}{\text{volume}}}{\rho g}\ \underbrace{d\bar{x}\cdot d\bar{a}}_{\text{volume}}/dt = \underbrace{\rho g\bar{v}}\cdot d\bar{a}$$

$\rho g\bar{v}$ represents a <u>flux density</u> of the property G;

we can now integrate over the desired surface S and get

$$\phi(G) = \int_S \rho g\bar{v}\cdot d\bar{a}$$

Let us recall that $\rho \frac{Dg}{Dt} = \frac{\partial}{\partial t}(\rho g) + \mathrm{div}(\rho g \bar{v})$

and now having $g = v_i$ (i.e. $G = \int_V \rho v_i \, d^3x = P_i$ = the total momentum within the volume V, i-th component)

$$\rho \frac{Dv_i}{Dt} = \frac{\partial}{\partial t}(\rho v_i) + \mathrm{div}(\rho v_i \bar{v}) \quad \text{or in tensor notation} \quad \rho \frac{Dv_i}{Dt} = \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_k}(\rho v_i v_k)$$

with Euler's equation

$$-\partial_i p - \rho \partial_i u = \partial_t(\rho v_i) + \partial_k(\rho v_i v_k)$$

that is (with $\partial_i p = \partial_k p \delta_{ik}$), $\partial_t(\rho v_i) = -\partial_k(\rho v_i v_k) - \partial_k p \delta_{ik} - \rho \partial_i u$

and thus $\underline{\partial_t(\rho v_i)} = -\partial_k(\rho v_i v_k + p\delta_{ik}) - \rho \partial_i u = \underline{-\partial_k \Pi_{ik} - \rho \partial_i u}$

If we now integrate this expression over a region R <u>fixed in time</u>, which allows us to say

$$\int_R \frac{\partial h}{\partial t} d^3x = \frac{d}{dt}\int_R h \, d^3x \,, \quad \text{so}$$

$$\frac{dP_i}{dt} = \frac{d}{dt}\int_R \rho v_i \, d^3x = \int_R \frac{\partial}{\partial t}(\rho v_i) d^3x = -\int_R \partial_k(\rho v_i v_k + p\delta_{ik}) d^3x - \int_R \rho \partial_i u \, d^3x = \text{with the divergence theorem}$$

$$= -\int_{\partial R} (\rho v_i v_k + p\delta_{ik}) n_k \, da - \int_R \rho \partial_i u \, d^3x = -\int_{\partial R} \Pi_{ik} n_k \, da - \int_R \rho \partial_i u \, d^3x$$

Here it becomes apparent that Π_{ik} is a momentum flux density (MOMENTUM FLUX DENSITY TENSOR), as this expression says that the rate of variation of momentum in a certain volume is due to the net flux of momentum escaping through the volume boundary, plus a volume force contribution if an external force field is present.

$\Pi_{ik} n_k \, da$ is the i-th component of the flux through da,

$\Pi_{ik} n_k$ the i-th component of the flux per unit area; explicitly,

$$\Pi_{ik} n_k = \rho v_i v_k n_k + p\delta_{ik} n_k = \rho v_i v_k n_k + p n_i \,,$$

the flux is made out of two contribution, the first being due to <u>advective transport</u> (momentum carried with the flow) and the second due to the exchange of contact forces (<u>pressure</u>).

If we take the generic direction indicated by a unit vector $\hat{n}$, in vector form we can see that the flux in that direction is

$$\underline{\underline{\Pi}} \cdot \hat{n} = \rho \bar{v}(\bar{v} \cdot \hat{n}) + p\hat{n} \qquad (\underline{\underline{\Pi}} \cdot \hat{n} \text{ is a contraction reducing order } 2 \to 1)$$

Now consider some specific directions

⊙ If $\hat{n} \parallel \bar{v}$, we observe the momentum flux along the flow and $\underline{\underline{\Pi}} \cdot \hat{n} = (\rho v^2 + p)\hat{n}$ (that is, there are both advective and pressure contributions)

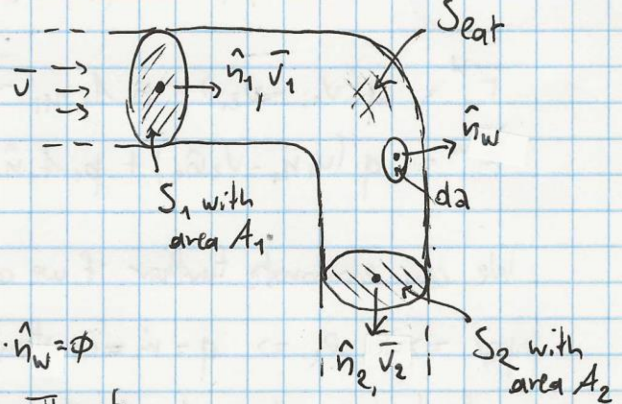
⊙ If $\hat{n} \perp \bar{v}$, $\underline{\underline{\Pi}} \cdot \hat{n} = p\hat{n}$ (that is, the transverse component is, perhaps trivially, only pressure-driven).

Finally, notice that momentum $\bar{P}$ is a vector, and its flux is an order 2 tensor; similarly, as energy is a scalar, we shall see soon that its flux is a vector (order 1 tensor).

Force on a pipe elbow

It is common observation to see that a pipe or hose making a bend experience a force against the curve when a fluid flows inside. We can use the momentum flux density tensor $\underline{\underline{\Pi}}$ to simplify the calculation of this force. Let us consider a piece of pipe as in the figure, under the hypotheses of ideal fluid, steady flow, and neglecting the influence of gravity.

Any portion of the pipe surface of area $d\bar{a}$ and outward normal unit vector $\hat{n}_W$ experiences a force exerted by the fluid



$d\bar{F}^W = p\hat{n}_W da$

while $\Pi_{ik} = p\delta_{ik} + \rho v_i v_k$ simplifies to

$\Pi_{ik} = p\delta_{ik}$ on the pipe wall because $\bar{v}\cdot\hat{n}_W = \phi$

(the fluid does not penetrate the solid wall). Therefore

$\Pi_{ik} n_{W_k} da = (p\delta_{ik} + \rho v_i v_k) n_{W_k} da = p\delta_{ik} n_{W_k} da = p n_{W_i} da = dF_i$

and the overall resultant force exerted by the fluid on the pipe reads

$$F_i^W = \int_{S_{lat}} p n_{W_i} da = \int_{S_{lat}} \Pi_{ik} n_{W_k} da \quad \text{with } S_{lat} \text{ pipe wall surface}$$

(notice that this simple expression exploits the fact that we can guarantee $\bar{v}\cdot\bar{n}_W = \phi$, which is true only at the pipe wall; it would not be possible for an ideal surface within the fluid).

Now let us recall the momentum flux conservation law

$\partial_t(\rho v_i) + \partial_k \Pi_{ik} = \phi$

which we shall integrate over the region R of the pipe elbow, defined by a boundary $\partial R = S_1 \cup S_2 \cup S_{lat}$:

$$\frac{dP_i}{dt} = \frac{d}{dt}\int_R \rho v_i d^3x = \underbrace{(+)}_{\text{since } \hat{n}_1 \text{ points inward}}\int_{S_1} \Pi_{ik} n_{1k} da - \int_{S_2} \Pi_{ik} n_{2k} da - \underbrace{\int_{S_{lat}} \Pi_{ik} n_{W_k} da}_{= F_i^W} \quad (*)$$

but for steady flow $\frac{d\bar{P}}{dt} = \phi$; now let us write Π_{ik} explicitly and consider an average, uniform value of p and $\bar{v}$ over each of the cross-sections S_1, S_2, $\to \bar{v}_1 // \hat{n}_1, \bar{v}_2 // \hat{n}_2$ so that, inverting $(*)$ for F_i^W we get

$$F_i^W = \left[\rho_1 v_{1i}(\bar{v}_1 \cdot \hat{n}_1) + p_1 n_{1i}\right] A_1 - \left[\rho_2 v_{2i}(\bar{v}_2 \cdot \hat{n}_2) + p_2 n_{2i}\right] A_2 =$$

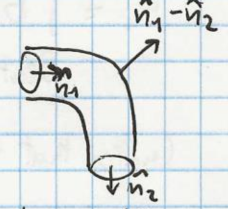
$$= (\rho_1 v_1 A_1) v_{1i} - (\rho_2 v_2 A_2) v_{2i} + p_1 A_1 n_{1i} - p_2 A_2 n_{2i}$$

The mass flow rate $q = \dot{m} = \int_S \rho \bar{v} \cdot d\bar{a} = \rho v A$ (for uniform ρ, v over S), i.e. the mass per unit time crossing a pipe cross section, is conserved for any cross section since the flow is steady, so $\rho_1 v_1 A_1 = \rho_2 v_2 A_2$ =>

$$F_i^W = q(v_{1i} - v_{2i}) + p_1 A_1 n_{1i} - p_2 A_2 n_{2i}$$, or in vector form,

$$\bar{F}^W = q(v_1 \hat{n}_1 - v_2 \hat{n}_2) + p_1 A_1 \hat{n}_1 - p_2 A_2 \hat{n}_2$$

We can elaborate further if we consider an <u>incompressible flow</u> (e.g., a water pipe) => $\rho_1 = \rho_2$ => $q = \dot{m} =$ constant implies $v_1 A_1 = v_2 A_2$, and a pipe with constant cross section $A = A_1 = A_2$. Now it is somewhat intuitive that for this case ($\rho_1 = \rho_2$, $A_1 = A_2$ => $v_1 = v_2$, steady flow with no dissipation) energy conservation requires that $p_1 = p_2$ too (formally, we shall verify this with Bernoulli's equation), i.e. pressure is also constant in value along the flow; the force $\bar{F}^W$ becomes



$$\bar{F}^W = \rho v^2 A(\hat{n}_1 - \hat{n}_2) + pA(\hat{n}_1 - \hat{n}_2) = (\rho v^2 + p) A (\hat{n}_1 - \hat{n}_2)$$

which tells us that $\bar{F}_W$ is pushing the pipe diagonally, towards the outside of the elbow. Notice that the same happens if we invert the orientation of the flow.

Let us throw in some numbers. Which term contributes the most to the force, the advective (kinetic) term ρv^2 or the pressure term? For instance, let us take water pipes for domestic use, where pressure ranges in 1.5-3 bar (1 bar = 10^5 Pa); $\rho_{H_2O} = 10^3\ kg/m^3$, so in order to compete with $p \sim 2 \cdot 10^5$ Pa, v must exceed 10 m/s. Simple observations will tell you that v is rather in the order of m/s, so the pressure term prevails.

Energy flux

As we did for momentum, we would like to investigate the rate of change of the energy stored in a region of space through which we observe the flow of an ideal fluid. In other words we shall see in what form does the fluid carry energy with its flow.

And once again a perfectly legitimate choice is following Landau's approach (well, if you dare to argue...); the sum of kinetic and internal energy per unit volume reads

$\frac{1}{2}\rho v^2 + \rho\varepsilon$ (with ε internal energy per unit mass);

said $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right)$ the time variation of the total energy, let us consider each term.

⊙ $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) = \frac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\bar{v}\cdot\frac{\partial \bar{v}}{\partial t}$$

Let us now plug in the continuity eq. $\frac{\partial\rho}{\partial t} = -\operatorname{div}(\rho\bar{v})$ and Euler's eq.* $\frac{\partial\bar{v}}{\partial t} = (-\bar{v}\cdot\operatorname{grad})\bar{v} - \frac{1}{\rho}\operatorname{grad}p$

so

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{1}{2}v^2\operatorname{div}(\rho\bar{v}) - \underbrace{\rho\bar{v}\cdot(\bar{v}\cdot\operatorname{grad})\bar{v}}_{\frac{1}{2}\bar{v}\cdot\operatorname{grad}(v^2)} - \bar{v}\cdot\operatorname{grad}p$$

To manipulate the last term on the right-hand side let us recall that

$$dw = Tds + \frac{1}{\rho}dp \Rightarrow \operatorname{grad}p = \rho\operatorname{grad}w - \rho T\operatorname{grad}s$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{1}{2}v^2\operatorname{div}(\rho\bar{v}) - \rho\bar{v}\cdot\operatorname{grad}\left(\frac{1}{2}v^2 + w\right) + \rho T\bar{v}\cdot\operatorname{grad}s$$

⊙ Since $\varepsilon + p/\rho = w$ enthalpy and $d\varepsilon = Tds - pdv = Tds + \frac{p}{\rho^2}d\rho$,

$$\Rightarrow d(\rho\varepsilon) = \varepsilon d\rho + \rho d\varepsilon = \varepsilon d\rho + \rho T ds + \frac{p}{\rho}d\rho = \left(\varepsilon + \frac{p}{\rho}\right)d\rho + \rho T ds = w d\rho + \rho T ds$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\varepsilon) = w\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} = -w\operatorname{div}(\rho\bar{v}) - \rho T\bar{v}\cdot\operatorname{grad}s$$

continuity eq. / adiabatic eq.: $\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + (\bar{v}\cdot\operatorname{grad})s = \phi$

Now we can sum up the two terms ($\rho T\bar{v}\cdot\operatorname{grad}s$ cancels out):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) = -\left(\frac{1}{2}v^2 + w\right)\operatorname{div}(\rho\bar{v}) - \rho\bar{v}\cdot\operatorname{grad}\left(\frac{1}{2}v^2 + w\right),$$ that is

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* = Consistently with the fact that we said total energy = kinetic + internal energy (no potential energy), in Euler's equation there is no potential from external force fields.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right) = -\mathrm{div}\left[\rho\bar{v}\left(\frac{1}{2}v^2+w\right)\right]$$

and by integration over a fixed region R and using the divergence theorem,

$$\frac{d}{dt}\int_R\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)d^3x = -\int_{\partial R}\rho\bar{v}\left(\frac{1}{2}v^2+w\right)\cdot d\bar{a}$$

i.e. $\left.\begin{array}{l}\text{variation of energy}\\ \text{(kinetic + internal)}\\ \text{stored in } R\end{array}\right\} = \left\{\begin{array}{l}\text{outgoing flux of kinetic energy + enthalpy}\\ \text{(not only internal energy } \varepsilon!\text{) carried by the fluid}\end{array}\right.$

or, by explicitly using $w=\varepsilon+p/\rho$,

$$\frac{d}{dt}\int_R\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)d^3x = -\int_{\partial R}\rho\bar{v}\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)\cdot d\bar{a} - \int_{\partial R}p\bar{v}\cdot d\bar{a}$$

i.e. variation of total energy = flux of total energy (advective term) + term yielded by pressure forces exerted by the rest of the fluid on the fluid within R

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Let us take another route (here I am following an approach found, e.g., in G. Parravicini's notes). Let us consider the forces on a fluid element; ignoring volume forces arising from external potentials (e.g., gravity), for an ideal fluid each infinitesimal surface element of a region R experiences a pressure force $\quad d\bar{F} = -p\hat{n}da$

and thus, if the velocity of the fluid (and of the surface element) is $\bar{v}$, the power is

$dP = -p\hat{n}\cdot\bar{v}da \quad$ so that the total power supplied to the fluid particle in R by the remaining fluid is

$$P = -\int_{\partial R}p\hat{n}\cdot\bar{v}da = -\int_R \mathrm{div}(p\bar{v})d^3x \underset{\uparrow}{\simeq} -\mathrm{div}(p\bar{v})V$$

with R infinitesimal and V = volume of R

If we divide by M = mass of the fluid in R we get the power per unit mass supplied to the fluid element $\dot{w}$ (if we call W the work, $\dot{W}$ = work per unit time = power P, $\dot{w}=\dot{W}/M$)

$$\dot{w} = -\mathrm{div}(p\bar{v})V/M = -\frac{1}{\rho}\mathrm{div}(p\bar{v})$$

Let us write the energy balance to the fluid particle in its motion (ideal flow without heat exchange). That is to say, we are writing the first law of thermodynamics for a system out of equilibrium, where kinetic energy can vary. The variation in total (kinetic + internal) energy must be balanced by the work performed by external forces; in terms of power per unit mass we have

$$\frac{D}{Dt}\left(\frac{1}{2}v^2\right) + \frac{D\mathcal{E}}{Dt} = \dot{W} = -\frac{1}{\rho}\,\mathrm{div}(p\bar{v})$$

Let us evaluate separately the rate of change of kinetic energy:

$$\frac{D}{Dt}\left(\frac{1}{2}v^2\right) = \bar{v}\cdot\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\,\bar{v}\cdot\mathrm{grad}\,p = -\frac{1}{\rho}\,\mathrm{div}(p\bar{v}) + \frac{1}{\rho}\,p\,\mathrm{div}\,\bar{v}$$

$$\mathrm{div}(p\bar{v}) = \bar{v}\cdot\mathrm{grad}\,p + p\,\mathrm{div}\,\bar{v}$$

$$\Rightarrow \underline{\frac{D\mathcal{E}}{Dt}} = -\frac{D}{Dt}\left(\frac{1}{2}v^2\right) - \frac{1}{\rho}\,\mathrm{div}(p\bar{v}) = \underline{-\frac{1}{\rho}\,p\,\mathrm{div}\,\bar{v}}$$

Notice here that since $D\mathcal{E}/Dt = -\frac{1}{\rho}p\,\mathrm{div}\,\bar{v}$, in an INCOMPRESSIBLE ideal flow ($\mathrm{div}\,\bar{v} = \emptyset$) <u>the internal energy is a constant of motion</u>; if the flow is steady $\mathcal{E}$ is constant along a streamline (as streamlines coincide with trajectories).

Recalling that for g quantity per unit mass $\rho\frac{Dg}{Dt} = \frac{\partial}{\partial t}(\rho g) + \mathrm{div}(\rho g\bar{v})$,

① $$\frac{D}{Dt}\left(\frac{1}{2}v^2+\mathcal{E}\right) = -\frac{1}{\rho}\,\mathrm{div}(p\bar{v}) \quad \text{implies}$$

$$\rho\frac{D}{Dt}\left(\frac{1}{2}v^2+\mathcal{E}\right) = \frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2+\mathcal{E}\right)\right] + \mathrm{div}\left[\rho\left(\frac{1}{2}v^2+\mathcal{E}\right)\bar{v}\right] = -\mathrm{div}\,p\bar{v}$$

that is

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2+\mathcal{E}\right)\right] = -\mathrm{div}\left[\left(\frac{1}{2}\rho v^2+\rho\mathcal{E}+p\right)\bar{v}\right] \underset{w=\mathcal{E}+p/\rho}{=} -\mathrm{div}\left[\rho\left(\frac{1}{2}v^2+w\right)\bar{v}\right]$$

<u>and integrating over a fixed region R</u>

$$\frac{d}{dt}\int_R\left(\frac{1}{2}\rho v^2+\rho\mathcal{E}\right)d^3x = -\int_{\partial R}\rho\bar{v}\left(\frac{1}{2}v^2+w\right)\cdot d\bar{a} = -\int_{\partial R}\rho\left(\frac{1}{2}v^2+\mathcal{E}\right)\bar{v}\cdot d\bar{a} - \int_{\partial R}p\bar{v}\cdot d\bar{a}$$

i.e. variation of total energy in R = ——— > = flux of total energy (advective term, energy carried by the fluid) + pressure forces → power supplied by the rest of the fluid

(as found equivalently by Landau)

② $$\rho\frac{D}{Dt}\left(\frac{1}{2}v^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \mathrm{div}\left(\frac{1}{2}\rho v^2\bar{v}\right) = -\mathrm{div}(p\bar{v}) + p\,\mathrm{div}\,\bar{v}$$

i.e. $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\mathrm{div}\left[\left(\frac{1}{2}\rho v^2+p\right)\bar{v}\right] + p\,\mathrm{div}\,\bar{v}$$

which we can once again integrate over R, region fixed in time, yielding

$$\frac{d}{dt}\int_R \left(\frac{1}{2}\rho v^2\right) d^3x = -\int_{\partial R} \frac{1}{2}\rho v^2 \bar{v}\cdot d\bar{a} - \int_{\partial R} p\bar{v}\cdot d\bar{a} + \int_R p\,\mathrm{div}\,\bar{v}\, d^3x$$

that is to say, the variation of kinetic energy within R is due to the advective flux of kinetic energy carried along by the flow + power supplied by pressure over ∂R + another term, $\int_R p\,\mathrm{div}\,\bar{v}\, d^3x$, <u>at the expense</u> of kinetic energy, that is to say some of the power does not contribute to the kinetic energy content. Notice indeed that if R is the whole fluid domain and if this volume is isolated from the rest of the universe, we can say that $\bar{v}\cdot\hat{n} = \phi$ on ∂R, $p = \phi$ on ∂R, $\Rightarrow$

$$\frac{d}{dt}\int_R \frac{1}{2}\rho v^2 d^3x = \int_R p\,\mathrm{div}\,\bar{v}\, d^3x$$: kinetic energy is not conserved in an isolated fluid system

③ $\rho\frac{DE}{Dt} = \frac{\partial}{\partial t}(\rho E) + \mathrm{div}(\rho E\bar{v}) = -p\,\mathrm{div}\,\bar{v}$ and again by integration over fixed R

$$\frac{d}{dt}\int_R \rho E\, d^3x = -\int_{\partial R} \rho E\bar{v}\cdot d\bar{a} - \int_R p\,\mathrm{div}\,\bar{v}\, d^3x$$

and here we can see that the time variation of internal energy is due to the flux term + a term that is the opposite of what we found in the rate of change of kinetic energy, thus showing us a <u>conversion</u> between the two types of energy. In an isolated fluid (as before)

$$\frac{d}{dt}\int_R \rho E\, d^3x = -\int_R p\,\mathrm{div}\,\bar{v}\, d^3x$$: internal energy is not conserved, either

Notes:
* The exchange between kinetic and internal energy can go both ways, and it is reversible since we are dealing with an ideal fluid;
* In the case (and <u>only</u> in the case) of incompressible flow, $\mathrm{div}\,\bar{v} = \phi$ so both kinetic and internal energy of the whole, isolated fluid are individually and separately conserved throughout the dynamics.

Energy flux in an external field

If the fluid is immersed in a volume force field $\bar{f}_v$ and we can define a potential u such that $\bar{f}_v = -\text{grad}\, u$ and is time-independent ($\partial u/\partial t = \phi$), there is an easy generalization for the expression of the energy flux.

This force $\bar{f}_v$ also supplies power to the fluid element under consideration:

$$P_v = \bar{f}_v \cdot \bar{v} = -\text{grad}\, u \cdot \bar{v} \underset{\substack{\uparrow \\ \text{since } \partial u/\partial t = \phi \text{ anyway}}}{=} -\left(\frac{\partial u}{\partial t} + \bar{v}\cdot\text{grad}\, u\right) = -\frac{Du}{Dt}$$

So that the energy balance(*) becomes

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + \varepsilon\right) = \underbrace{-\frac{1}{\rho}\,\text{div}(p\bar{v})}_{\substack{\text{power supplied by the} \\ \text{pressure exerted by the} \\ \text{rest of the fluid}}} \underbrace{-\frac{Du}{Dt}}_{\substack{\text{power supplied by the \underline{conservative}} \\ \text{volume force field}}} \quad \Rightarrow$$

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + \varepsilon + u\right) = -\frac{1}{\rho}\,\text{div}(p\bar{v}) \quad \text{and exploiting the expression } \rho\frac{Dg}{Dt} = \frac{\partial(\rho g)}{\partial t} + \text{div}(\rho g\bar{v})$$

$$\Rightarrow \underline{\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + \varepsilon + u\right)\right]} = -\text{div}\left[\rho\left(\frac{1}{2}v^2 + \varepsilon + u\right)\bar{v}\right] - \text{div}(p\bar{v}) = \underline{-\text{div}\left[\rho\left(\frac{1}{2}v^2 + w + u\right)\bar{v}\right]}$$

We can also define the <u>mechanical energy</u> $\varepsilon_m = \frac{1}{2}v^2 + u$ (sum of kinetic and potential energy)

$$\Rightarrow \frac{\partial}{\partial t}\left[\rho(\varepsilon_m + \varepsilon)\right] = -\text{div}\left[\rho(\varepsilon_m + w)\bar{v}\right] \quad \text{and by integration over a fixed region } R,$$

$$\frac{d}{dt}\int_R \rho(\varepsilon_m + \varepsilon)\,d^3x = -\int_{\partial R} \rho(\varepsilon_m + w)\bar{v}\cdot d\bar{a} = -\int_{\partial R} \rho(\varepsilon_m + \varepsilon)\bar{v}\cdot d\bar{a} - \int_{\partial R} p\bar{v}\cdot d\bar{a}$$

where the considerations already expressed in the absence of $\bar{f}_v$ still hold but now we must consider the <u>mechanical energy</u> as a whole (with the remarks about conservation and conversion into internal energy) <u>instead of the kinetic energy.</u>

(*) = now the balance reads: The work performed by the outside world onto the fluid element, once subtracted of the heat released by the fluid to the outside, equals the increment in mechanical (kinetic + potential) energy and internal energy of the fluid element (general form of non-equilibrium first law of thermodynamics).