Bernoulli's equation

Bernoulli's equation expresses a conservation property that holds for a steady flow of an ideal fluid, i.e. the velocity field is constant in each point of the fluid domain, that is when $\bar{v}(\bar{x},t) \to \bar{v}(\bar{x})$ (remember that then streamlines coincide with trajectories).

Landau's derivation of Bernoulli's equation considers Euler's equation for an isentropic flow in order to be able to write $-\frac{1}{\rho}\,\mathrm{grad}\,p \to -\mathrm{grad}\,w$ and easily proceed from there. As a matter of fact this is not required, and we can work out our way from Euler's equation in its most general form

$$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\,\mathrm{grad}\,p - \mathrm{grad}\,u \qquad \text{and perform a scalar product times } \bar{v} \Rightarrow$$

$$\bar{v}\cdot\frac{D\bar{v}}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}v^2\right) = -\bar{v}\cdot\frac{1}{\rho}\,\mathrm{grad}\,p - \bar{v}\cdot\mathrm{grad}\,u$$

Recall $dw = T\,ds + \frac{1}{\rho}dp \;\rightsquigarrow\; \mathrm{grad}\,w = T\,\mathrm{grad}\,s + \frac{1}{\rho}\,\mathrm{grad}\,p$

and the adiabatic equation $\frac{\partial s}{\partial t} + \bar{v}\cdot\mathrm{grad}\,s = \phi$ and plug them into

$$\frac{1}{\rho}\bar{v}\cdot\mathrm{grad}\,p = \bar{v}\cdot\mathrm{grad}\,w - T\bar{v}\cdot\mathrm{grad}\,s = \bar{v}\cdot\mathrm{grad}\,w + T\frac{\partial s}{\partial t}$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{1}{2}v^2\right) = -\bar{v}\cdot\mathrm{grad}\,w - T\frac{\partial s}{\partial t} - \bar{v}\cdot\mathrm{grad}\,u$$

that is $$\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) = -\bar{v}\cdot\mathrm{grad}\left(\frac{1}{2}v^2 + w + u\right) - T\frac{\partial s}{\partial t}$$

Now let us ask for steady flow; therefore $\partial/\partial t = \phi$ and we get

$$\bar{v}\cdot\mathrm{grad}\left(\frac{1}{2}v^2 + w + u\right) = \phi$$

Remember that $df = \mathrm{grad}\,f\cdot d\bar{l}$ which expresses the relationship between the gradient and the directional derivative along the direction of a displacement $d\bar{l}$; if we consider $d\bar{l} \parallel \bar{v}$ we have

$$\bar{v}\cdot\mathrm{grad}\,f = v\,df/dl$$

$$\Rightarrow v\frac{d}{dl}\left(\frac{1}{2}v^2 + w + u\right) = \phi \quad \text{along a streamline, that is}$$

$$\boxed{\frac{1}{2}v^2 + w + u = \text{constant}}$$ along a streamline (with different constant value $\forall$ streamline)

Bernoulli's equation

with equivalent formulations

$$\frac{1}{2}v^2 + \varepsilon + \frac{p}{\rho} + u = \text{constant}$$

$$\mathcal{E}_m + w = \text{constant}$$

$$\frac{1}{2}v^2 + \varepsilon + \frac{p}{\rho} + gz = \text{constant} \quad \left[\text{when } \bar{g} = -\operatorname{grad} u = -g\hat{e}_z \Rightarrow u = gz\right]$$

One can also see how Bernoulli's equation stems from the conservation of energy by means of an alternative demonstration. Conservation of energy being written as

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + \varepsilon + u\right) = -\operatorname{div}\left[\rho\left(\frac{1}{2}v^2 + w + u\right)\bar{v}\right]$$

for steady flow $\partial_t(\mathcal{E}_m + \varepsilon) = \phi \Rightarrow \operatorname{div}\left[\rho\left(\frac{1}{2}v^2 + w + u\right)\bar{v}\right] = \phi$

and taking the integral over a region R and using the divergence theorem

$$\int_{\partial R} \rho\left(\frac{1}{2}v^2 + w + u\right)\bar{v}\cdot\hat{n}\, da = \phi$$

Now let us consider a portion of a streamtube as R; the lateral surface of the tube does not contribute to the integral, as the flow crosses only S_1, S_2 entrance and exit surfaces of the tube. Shrinking the tube to an infinitesimal cross section, such that it encloses a single streamline, quantities in the integrand are uniform over S_1, S_2; so the integral becomes

$$-\rho_1(\mathcal{E}_{m1} + w_1)\bar{v}_1\cdot\hat{n}_1\, dA_1 + \rho_2(\mathcal{E}_{m2} + w_2)\bar{v}_2\cdot\hat{n}_2\, dA_2 = \phi;$$

coupling this with the conservation of the mass flow rate

$$-\rho_1\bar{v}_1\cdot\hat{n}_1\, dA_1 + \rho_2\bar{v}_2\cdot\hat{n}_2\, dA_2 = \phi$$

we get $\quad \mathcal{E}_{m1} + w_1 = \mathcal{E}_{m2} w_2 \Rightarrow \underline{\mathcal{E}_m + w = \text{constant}}$

that is $\boxed{\frac{1}{2}v^2 + w + u = \text{constant}}$ along the streamline

A few significant remarks.

⊙ <u>Incompressible flow</u>

We saw that for an ideal fluid $\frac{D\varepsilon}{Dt} = -\frac{1}{\rho}p\operatorname{div}\bar{v}$

and thus when the flow is incompressible, $\Rightarrow \operatorname{div}\bar{v} = \phi$, internal energy ε turns out to be conserved in the flow (a constant of motion), that is to say, ε is constant along a streamline. Hence Bernoulli's equation

$$\frac{1}{2}v^2 + \mathcal{E} + \frac{p}{\rho} + u = \text{constant}$$

reduces to $\boxed{\frac{1}{2}v^2 + \frac{p}{\rho} + u = \text{constant}}$ along a streamline.

⊙ Real (viscous) fluid

While for an ideal fluid we have a conservation law $\frac{D}{Dt}\left(\frac{1}{2}v^2 + w + u\right) = \phi$,

along the flow of a real fluid losses occur due to friction; said $\dot{W}_f$ the power dissipation per unit mass,

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + w + u\right) = -\dot{W}_f < \phi$$

⊙ Bernoulli's equation as we wrote it so far states a conservation law in terms of energy (per unit mass). When $u = gz$, dividing by g (in the incompressible flow ,

$$\boxed{\frac{v^2}{2g} + \frac{p}{\rho g} + z = \text{constant}}$$

i.e. the law is expressed in terms of a height or HEAD: The sum of

$\frac{v^2}{2g}$ = KINETIC HEAD, $\frac{p}{\rho g}$ = PRESSURE HEAD, z = ELEVATION HEAD = TOTAL HEAD

is constant.

The sum of pressure and elevation head is also called PIEZOMETRIC or HYDRAULIC head.

Similarly, one could write Bernoulli's equation in dimensions of a pressure:

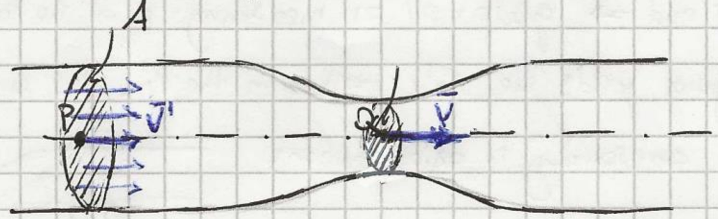
$\boxed{\frac{1}{2}\rho v^2 + p + \rho g z = \text{constant}}$ and the terms are called

$$\left.\begin{aligned} \tfrac{1}{2}\rho v^2 &= \text{DYNAMIC PRESSURE} \\ p &= \text{STATIC PRESSURE} \\ \rho g z &= \text{ELEVATION PRESSURE} \end{aligned}\right\} \text{ sum = TOTAL PRESSURE = constant}$$

We shall see some examples of physical situations that can be explained by means of Bernoulli's equation. We will always make use of the additional hypothesis of incompressibility. When the flow is compressible or viscosity occurs, basic features and interpretations qualitatively remain the same, with suitable corrections to account for compressibility and/or viscous losses.

Venturi effect and Venturimeter (Venturi tube)

Consider a pipe having a horizontal longitudinal symmetry axis (such that we can ignore elevation effects in Bernoulli's equation), just for maximum simplicity. The flow in the tube passes from a cross section A (point P on the axis) to a constricted cross section (point Q) $a < A$ (see figure). In a steady and incompressible flow,



⊙ by mass flow rate conservation between P (area A, velocity $\bar{v}'$) and Q (area a, velocity $\bar{v}$),

$$v'A = va \Rightarrow v = \frac{A}{a}v'$$ speed increases in the constriction

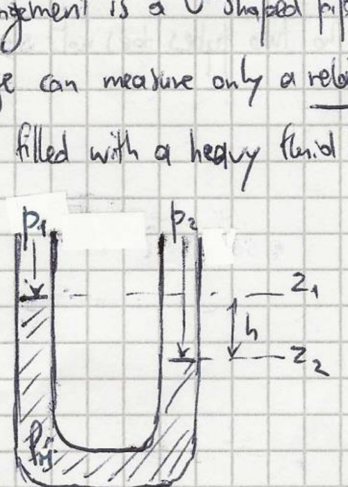
⊙ by Bernoulli's equation applied on the streamline connecting P and Q,

$$\frac{1}{2}v'^2 + \frac{1}{\rho}p(P) = \frac{1}{2}v^2 + \frac{1}{\rho}p(Q) \qquad (z(P) = z(Q))$$

$$\Rightarrow p(Q) - p(P) = \frac{1}{2}\rho(v'^2 - v^2) = \frac{1}{2}\rho v'^2\left[1 - (A/a)^2\right] < 0$$

pressure decreases in the constriction (a somewhat counterintuitive effect called "hydrodynamic paradox").

If we could measure the difference in static pressure $p(P) - p(Q)$ we could invert the expression and obtain the flow speed v' and mass flow rate $v'A$. This is indeed possible by exploiting the principle of the MANOMETER. The simplest manometer arrangement is a U-shaped pipe whose ends are at the two pressure to be measured (this gauge can measure only a relative pressure, i.e. a p difference) p_1 and p_2. The tube is partially filled with a heavy fluid (i.e., $\rho_M > \rho_F$ density of the fluid where p is to be measured).

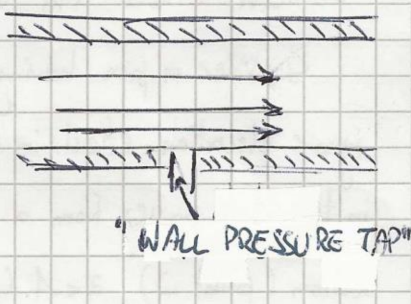


In hydrostatic equilibrium

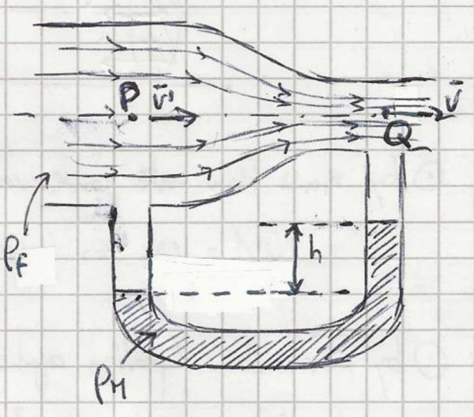
$$\frac{p_1}{\rho_M} + gz_1 = \frac{p_2}{\rho_M} + gz_2 \Rightarrow p_2 - p_1 = \rho_M g(z_1 - z_2) = \rho_M gh$$

and measuring the elevation difference yields the pressure difference.

This technique can be applied to the Venturi tube, but how? How do we measure the static pressure in a pipe where there is a fluid flow?



If we make a clean, perpendicular hole in the wall of a pipe, in a region with straight streamlines, parallel to the wall, we can say that there is no pressure variation normal to the straight streamlines (ignoring a negligible gravity effect, $D\bar{v}/Dt = \phi$ $\perp$ streamlines and $\Rightarrow$ $\mathrm{grad}\, p = \phi$) $\Rightarrow$ measuring p at the tap with a manometer or any pressure gauge yields the static pressure in the flow. We can place for instance a differential manometer corresponding to axial positions P and Q and get



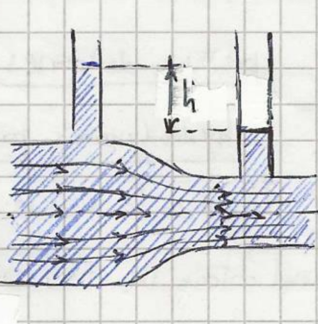
$$v' = \left(2[p(P) - p(Q)] \Big/ \rho_f[(A/a)^2 - 1] \right)^{\frac{1}{2}}$$

and $p(P) - p(Q) = \rho_M g h$

or, if we want to be more precise and not neglect the fact that in the left arm of the manometer there is a column h of lighter fluid ρ_f, $p(P) - p(Q) = (\rho_M - \rho_f) g h$ $\Rightarrow$

$$v' = \left(\frac{2gh(\rho_M - \rho_f)}{\rho_f[(A/a)^2 - 1]} \right)^{\frac{1}{2}}$$

Equivalently, one may think of an arrangement like in the figure $\rightarrow$ (where again the fluid in the vertical pipes is brought to rest, but this time by equilibration with raising to a greater elevation).



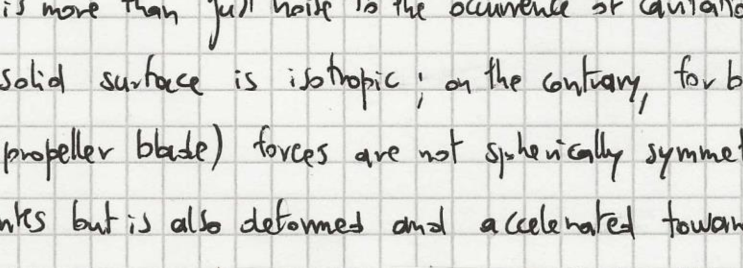
Cavitation

The Venturi tube is a good example of a situation where pressure can undergo significant, even drastic and abrupt variations. Indeed p decreases in the constriction and then increases again when the pipe enlarges after it.

In a liquid, bubbles are often present (a phenomenon helped by the presence of impurities, or pipe surface roughness, and undissolved gases, - sources of bubble nucleation) and pressure reduction regions will make them grow in size, as they have to be in equilibrium with the liquid's vapour pressure. This is what can happen in the Venturi's constriction. Then as soon as the flow experiences the pressure increase after the constriction (with the pipe's cross section growing again) the bubbles are compressed and collapse. This process is fast as it follows the flow, so this expansion and collapse (CAVITATION) generates acoustic waves, that is the noise sometimes heard in pipes (that sort of rattling like gravel moving around in a container).

This phenomenon, if not carefully avoided, can occur in a variety of situations, e.g., in marine propellers, turbines, pumps and in turbomachines in general, where motion creates pressure gradients. The well-known phenomenon of "singing propellers" (cavitation noise occurring in submarine propellers, exploited sometimes for tactical uses) is an example of cavitation.

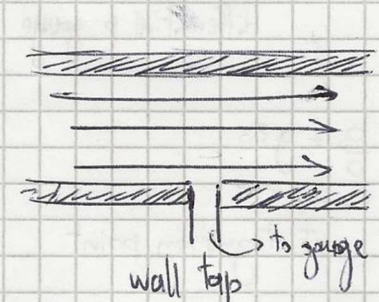
There is more than just noise to the occurrence of cavitation. The collapse of bubbles far from a solid surface is isotropic; on the contrary, for bubbles close to a surface (pipe wall or propeller blade) forces are not spherically symmetric, so that the bubble not only shrinks but is also deformed and accelerated towards the solid surface; the liquid following it is also accelerated (liquid jet) and hits the surface at high speed, generating microscopic corrosion events; a prolonged exposure to such phenomenon can give rise to significant damage.

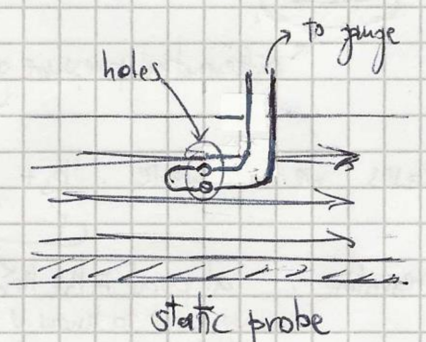


Asymmetric collapse of a bubble close to the surface and formation of a high-speed liquid jet impacting the surface

Static probe and pitot tube

We have already seen that static pressure in a closed pipe can be accessed using a "wall tap", i.e. a hole perpendicular to the flow, and connecting there a pressure gauge. One can also measure pressure far from the wall, or in an open basin, or where streamlines are curved, by properly designing and placing a STATIC PRESSURE PROBE. A static probe consists in a pipe inserted in such a way that its tip faces the flow; the tip itself is closed, but small holes on the sides allow fluid communication in such a way that at equilibrium fluid is at rest inside the probe stem and the pressure is indeed equal to the static pressure in the flow.





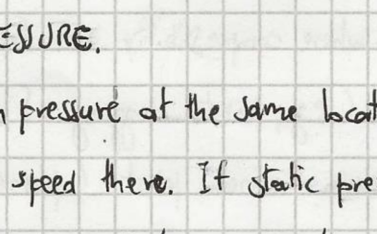
Now let us imagine we can find a point in the flow where the fluid is brought to zero speed (without friction). Since we can always invoke Bernoulli's equation along a streamline, we can state the conservation of total pressure between a point upstream, where $v \neq 0$, and this point at null velocity (STAGNATION POINT):

$$\frac{1}{2}v^2 + \frac{p}{\rho} = \frac{p_0}{\rho} \quad \text{(considering a horizontal streamline or negligible elevation effect)}$$

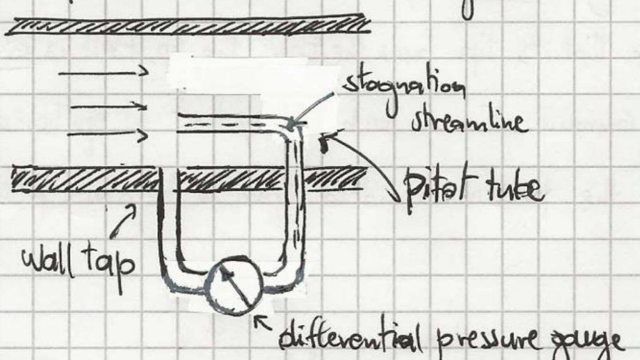
$$\Rightarrow p_0 = p_p + \frac{1}{2}\rho v^2 \text{ is called STAGNATION PRESSURE.}$$

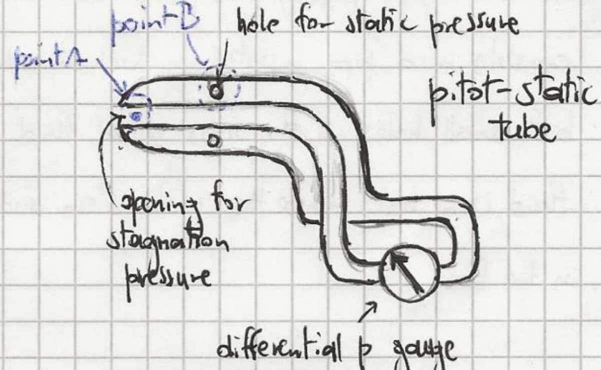
If we could measure both static and stagnation pressure at the same location (or at very close points), then we could obtain the flow speed there. If static pressure can be measured with a wall tap or a static probe, a device that measures the stagnation pressure is the PITOT TUBE (after its original inventor, Henri Pitot, 1695-1771).

The pitot tube resembles the static probe, but its tip facing the flow is open; in a steady flow state, the point O at the entrance must be at rest, i.e. a stagnation point with pressure p_0 and speed $V_0 = 0$.



We can combine the stagnation pressure with a static pressure measurement by using a wall tap (if the stagnation point is not too far) or by using a <u>pitot-static tube</u>, i.e. a device comprising two coaxial pipes — an inner pipe acting as pitot tube, and an outer pipe (with closed entrance and side ports) to add the static pressure measurement (see figures).





In both cases we can write $\underbrace{\frac{p_s}{\rho} + \frac{1}{2}v^2 + gz_s}_{\text{at point in the flow where we want to know } v \text{ and measure static pressure } p} = \underbrace{\frac{p_0}{\rho} + gz_0}_{\text{at stagnation point}}$

along a streamline :

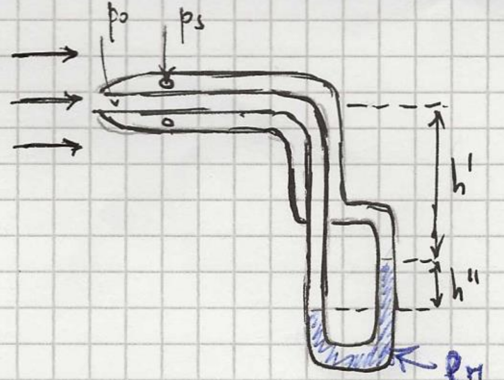
$$\Rightarrow \quad v = \left[\frac{2(p_0 - p_s)}{\rho} + 2g(z_0 - z_s)\right]^{\frac{1}{2}} \simeq \left[\frac{2(p_0 - p_s)}{\rho}\right]^{\frac{1}{2}}$$

for negligible elevation effect

Note that points A and B are not coincident but we are assuming static pressures $p_A = p_B$; so pressure variations streamwise must be negligible to get a correct result.

Pitot-static probes are commonly used in a variety of applications, such as race cars and airplanes (where compressibility and friction effects have to be properly compensated). Probe malfunctions (e.g., due to clogging of apertures) and maintenance errors have caused dramatic accidents even in the near past (plane crashes).

Example 1: A pitot-static system is placed facing an air flow to determine the air speed. A U-shaped manometer used as differential pressure gauge reads a pressure difference $h'' = 30$ mm of mercury between pitot and static ports. What is the air flow speed (air is at standard conditions, so $\rho_a = 1.23$ Kg/m^3; for mercury, $\rho_H = 13.6 \cdot 10^3$ Kg/m^3) ?



Bernoulli's eq. along the stagnation streamline:

$$\frac{p_0}{\rho_a} = \frac{p_s}{\rho_a} + \frac{v^2}{2}$$ (p_0 stagnation p, p_s static p)

$$\Rightarrow v = \sqrt{2(p_0-p_s)/\rho_a}$$

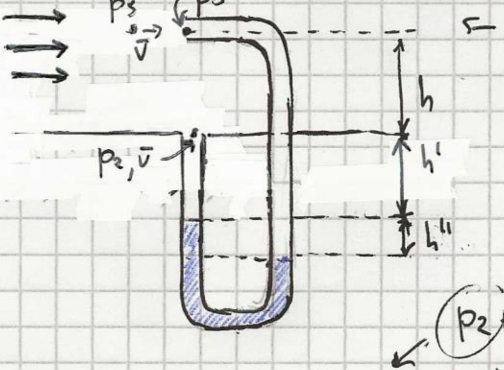
Hydrostatic equilibrium in the manometer:

$$p_0 + \rho_a g h' + \rho_a g h'' = p_s + \cancel{\rho_a g h'} + \rho_H g h''$$

$$\Rightarrow p_0 - p_s = (\rho_H - \rho_a) g h''$$ ← insert in expression for v ⇒

$$v = \sqrt{2(\rho_H-\rho_a)gh''/\rho_a} \; \left(\approx \sqrt{2\rho_H g h''/\rho_a}\right) = 80.67 \text{ m/s}$$

Example 2: Now imagine we are measuring the same air flow, but we are using a combination of simple pitot tube (without coaxial static port) and wall tap (see figure), with the manometer yielding the same reading. Compare the result.



← Bernoulli's eq. along the stagnation streamline:

$$\frac{p_0}{\rho_a} = \frac{p_s}{\rho_a} + \frac{v^2}{2} \Rightarrow v = \sqrt{2(p_0-p_s)/\rho_a}$$

Hydrostatic equilibrium states:

$$p_2 + \rho_a g h' + \rho_H g h'' = p_0 + \rho_a g h + \rho_a g h' + \rho_a g h''$$

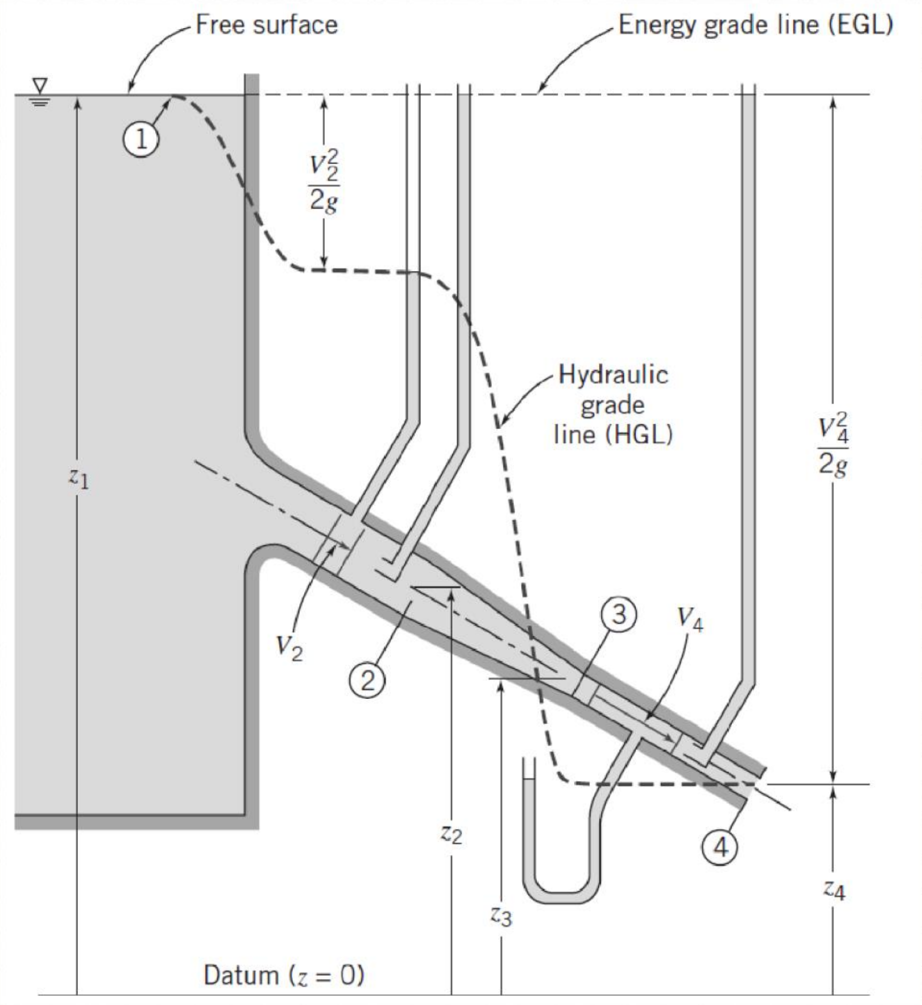
$$p_2 = p_s + \rho_a g h$$

$$\Rightarrow p_s + \rho_H g h'' = p_0 + \rho_a g h''$$

$$\Rightarrow p_0 - p_s = (\rho_H - \rho_a) g h''$$

hence $v = \sqrt{2(\rho_H-\rho_a)gh''/\rho_a}$

exactly as the measurement done with the pitot-static system. So the fact that the pitot (total head) and tap (static head) measurements differ in height by h has no actual influence.

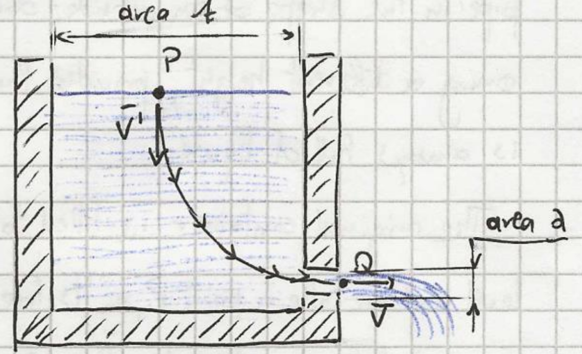


Source: "Fox and McDonald's Introduction to Fluid Mechanics" (10th edition), John W. Mitchell (Wiley, 2020)

The concepts of total head $p/\rho g + v^2/2g + z$ (= constant) and hydraulic head can also be illustrated graphically plotting them as in the figure above along the flow; here they are equivalently called ENERGY GRADE LINE, which is indeed the total head and is constant for frictionless (ideal) flow, and HYDRAULIC GRADE LINE, i.e. the hydraulic (pressure + elevation) head. The latter drops in the example as kinetic energy is gained along the streamline from left to right.

Torricelli's law

This result obtained by Evangelista Torricelli could later be reinterpreted as as a special case of Bernoulli's equation. We consider here a tank whose horizontal cross section is A and make a hole in the lower part of the tank's lateral wall, with aperture area $a \ll A$. The tank is filled with a liquid (e.g. water) to a height h with respect to the position of the hole. By assuming <u>steady and incompressible flow</u> we can determine the efflux speed of the liquid at the hole.



Indeed, since the fluid cannot penetrate the tank wall, v_n (normal component of $\bar{v}$) at the wall is zero, the walls of the tank and drain are a stream/flux tube. Calling $\bar{v}', \bar{v}$ the velocities at the upper free surface and drain hole, respectively, the conservation of mass flow rate for steady incompressible flow states $\quad v'A = va \quad \Rightarrow \quad v' = \frac{a}{A} v$ which is negligible for $a \ll A$

Now let us consider a streamline from the upper free surface (point P) to the open drain (point Q) and let us apply Bernoulli's equation between P and Q:

$$\frac{1}{2} v'^2 + \frac{1}{\rho} p(P) + g z(P) = \frac{1}{2} v^2 + \frac{1}{\rho} p(Q) + g z(Q)$$

and with $v' = 0$, $p(P) = p_{atm}$ and $p(Q) = p_{atm}$, too just outside the hole, we get

$$\frac{1}{2} v^2 = g\left[z(P) - z(Q)\right] = gh$$

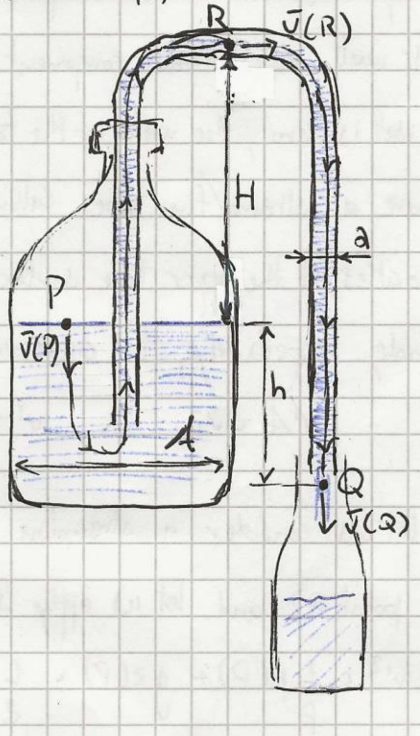
$$\Rightarrow \boxed{v = \sqrt{2gh}} \quad \text{Torricelli's law}$$

This law tells us that the efflux speed is the same as the one that would by acquired by free fall from the height h. This does not reflect perfectly what happens in real life: Unless the drain is a capillary, the flow is not really that regular (laminar, i.e. the velocity field is made of "well-ordered", parallel streamlines) but easily becomes turbulent. Yet the mass flow rate predicted by Torricelli's law is still approximately correct:

$$q = va = a\sqrt{2gh}$$

Siphon

Have you ever bottled wine previously contained in a demijohn? Such experience (made memorable by a non conservation principle according to which, the wine volume in the bottles at the end is rarely the same as it was in the original container, and no one knows what happened to the missing volume) is another consequence of Bernoulli's eq., and in particular of the working principle of a SIPHON. A Siphon is any type of pipe in the shape of an upside-down U, which allows one to transfer a liquid overcoming a height difference, provided that the exit is lower than the entrance and the pipe is always full of liquid.



The original container is filled to a certain level and we take a point P on its free surface; we shall follow a streamline from P through a positive height difference H up to the maximum height in the pipe (point R) and down to the exit (point Q) at a lower height with respect to P ($z(P) - z(Q) = h$). Since P and Q are "in the open", $p(P) = p(Q) = p_{atm}$; since the horizontal cross section A of the container is $\gg$ than the pipe cross section a, the level in the container decreases very slowly due to conservation of the mass flow rate ($v(P)A = v(Q)a \Rightarrow v(P) = v(Q)\, a/A \to 0$), hence by writing Bernoulli's equation

$$\cancel{\frac{1}{2}v^2(P)} + \cancel{\frac{1}{\rho}p_{atm}} + g\,z(P) = \frac{1}{2}v^2(Q) + \cancel{\frac{1}{\rho}p_{atm}} + g\,z(Q)$$

where we have the usual assumptions of steady and incompressible ideal flow;

$$\Rightarrow \frac{1}{2}v^2(Q) = g\left[z(P) - z(Q)\right] = gh \quad \Rightarrow \quad v = \sqrt{2gh} \quad \text{with mass flow rate } q = a\sqrt{2gh}$$

exactly like in Torricelli's law. Notice that $h > 0$ is required, i.e. the exit of the pipe has to be lower than the liquid height in the container; in order to stop the flow, it is enough to raise the end of the pipe with the bottle.

Is there a limit to the maximum height difference H that can be overcome? Let us discover it by applying Bernoulli's eq. between P and R:

$$\frac{1}{2}\cancel{v^2(P)} + \frac{1}{\rho} p_{atm} + g z(P) = \frac{1}{2} v^2(R) + \frac{1}{\rho} p(R) + g z(R)$$

where we require $p(R)$ to be positive ($p<\phi$ is physically meaningless):

$$p(R) = p_{atm} + \rho g [z(P) - z(R)] - \frac{1}{2} v^2(R) = p_{atm} - \rho g H - \frac{1}{2} v^2(R) \geq \phi$$

$$\Rightarrow \rho g H \leq p_{atm} - \frac{1}{2} v^2(R)$$

and the maximum value is obtained when the fluid just stops when it reaches R, i.e. $v(R)=\phi \Rightarrow$

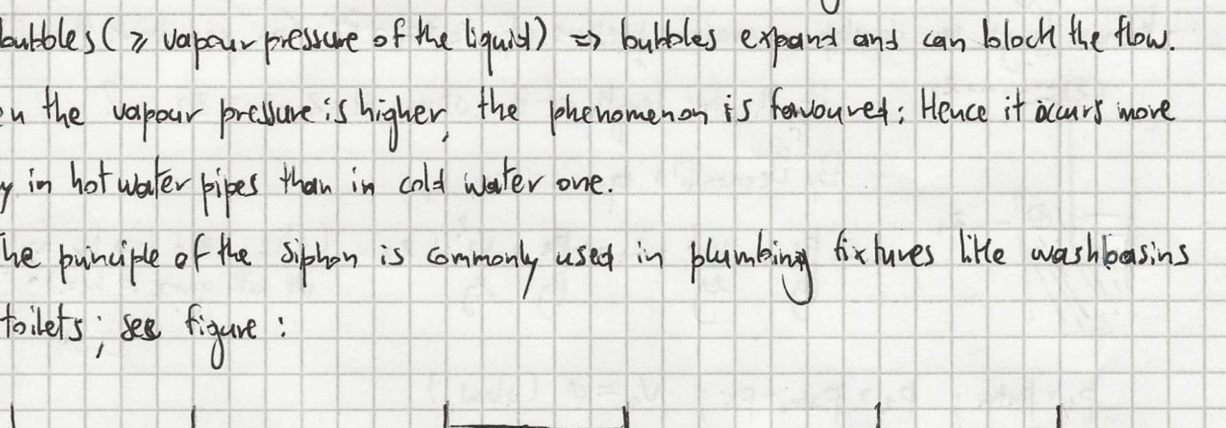
$$\boxed{H_{max} = p_{atm}/\rho g}$$

i.e. the maximum height difference is equal to the atmospheric pressure head

For water, $H_{max} \simeq 10\,m$; this is also why a typical centrifugal pump used, e.g., to pump water from a well or acquifer in agriculture for field irrigation, cannot bring up water from a depth greater than 10 m (more realistically, ~ 7-8 m) and a submersible pump must be installed at the bottom to push up water from greater depths.(*)

The presence of undissolved gases and thus the formation of gas bubbles can stop the flow at heights < H_{max}, because the pressure in the pipe decreases, as we have seen (it is a constriction, by the way: see Venturi tube), and it does no longer balance the pressure inside the bubbles ($\gtrsim$ vapour pressure of the liquid) $\Rightarrow$ bubbles expand and can block the flow. When the vapour pressure is higher, the phenomenon is favoured: Hence it occurs more easily in hot water pipes than in cold water one.

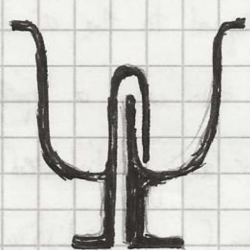
The principle of the siphon is commonly used in plumbing fixtures like washbasins and toilets; see figure:



Static situation; no communication (flow) between basin and drain	The basin gets filled the height is above the U and the flow can begin	The basin keeps emptying until the basin level is higher than that of the drain

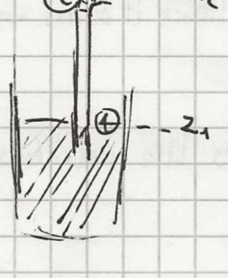
Siphons can occur in nature, e.g. in caves, and can be dangerous because flooding can occur putting at severe risk the life of cave explorers.

Another example of more entertaining and lighter nature is the Pythagorean cup (or Pythagoras cup, or greedy cup), credited to philosopher Pythagoras of Samos (... and now a popular tourist's souvenir in the island of Samos). The column in the center of the cup has an inner chamber shaped as a siphon going straight to the bottom of the cup's stem; so, when filled without moderation up to a level beyond the central column, the siphon fills up and starts draining the fluid until it is empty. Legend goes that Pythagoras invented this cup to limit the consumption of wine by workers building the Eupalinian aqueduct in Samos.



Cross section of a Pythagorean cup

(*) = Let us play a game. Have a chocolate milkshake at the mall and try to drink it by sucking it with a very long straw. Consider that $\rho = 1200 \ kg/m^3$ and that suction by human lungs can develop a vacuum pressure $p_s \simeq 3000$ Pa. Can you drink the milkshake if the free length of the straw $H = z_2 - z_1 = 30$ cm?



By Bernoulli's eq.

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$ (+h_f head loss by friction; we will ignore it here)

$p_1 = p_{atm}$; $p_2 = p_{atm} - p_s$; $V_1 \simeq \phi$ (slow!)

$$\frac{V_2^2}{2g} = \frac{p_s}{\rho g} - H = \frac{3000}{1200 \cdot 9.81} - 0.3 = 0.255 - 0.3 = -0.005 < \phi$$ impossible!

($-h_f$, too, to be true)

The straw must have a length above the free surface < 25.5 cm.

Lift – what's the difference between a spoon and an airplane?

A spoon in a water jet

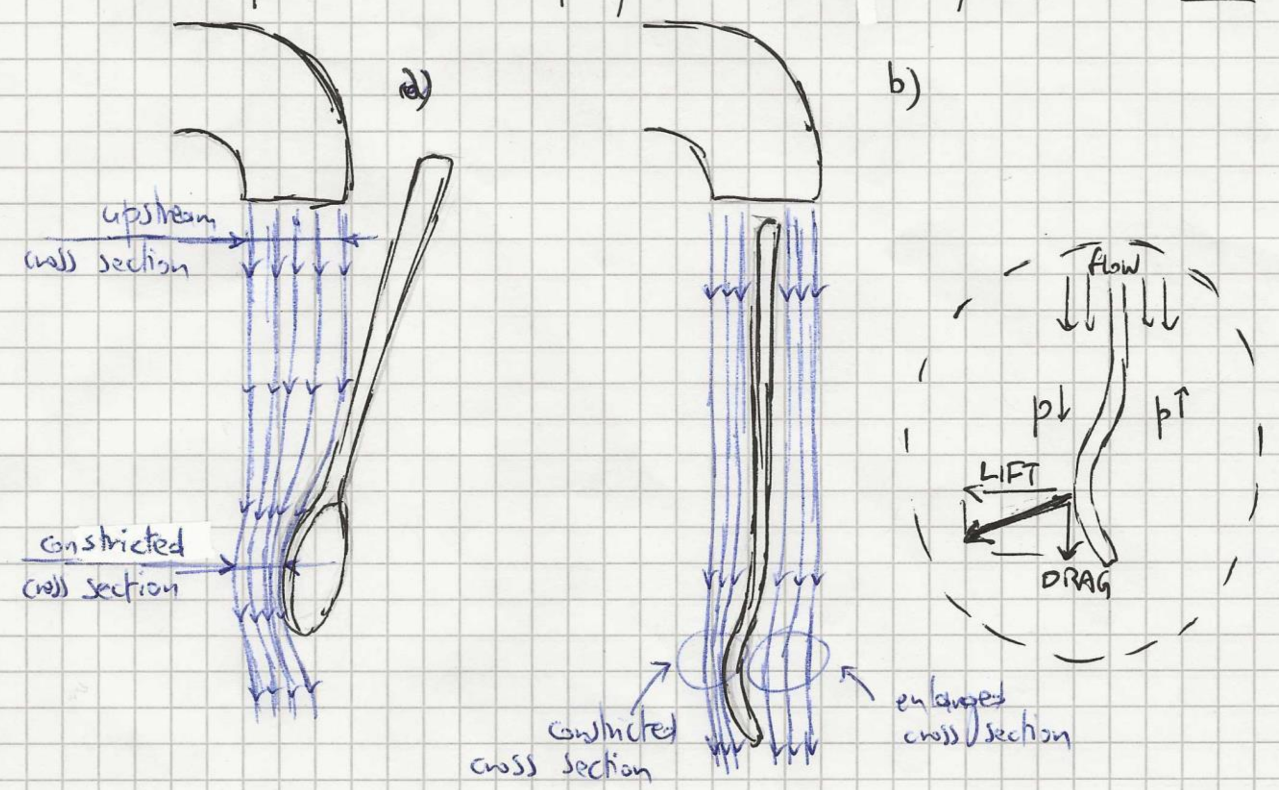
If you bring a spoon into contact with the water jet from a faucet, in such a way that the convex surface of the spoon is the one that is wetted first, you can clearly see and feel that the spoon is not pushed back, but pulled into the flow. This happens because the insertion of the spoon reduces the cross section of the streamtube upstream of the spoon $\Rightarrow$ by Bernoulli's eq. speed increases and pressure decreases locally on the spoon's surface; at the same time, what happens to the concave face of the spoon?

a) If it is still outside of the jet, $p = p_{atm} > p$ on the convex face;

b) If it is inside the jet, here the flux tube is enlarged $\Rightarrow$ once again speed is reduced and pressure increased.

In both cases there is a net Δp across the spoon's sides $\Rightarrow$ a force is created

* with a component NORMAL to the spoon (possibly the main component): <u>LIFT</u>;
* with a component ALONG the spoon, and oriented downstream, i.e. a resistance*: <u>DRAG</u>.



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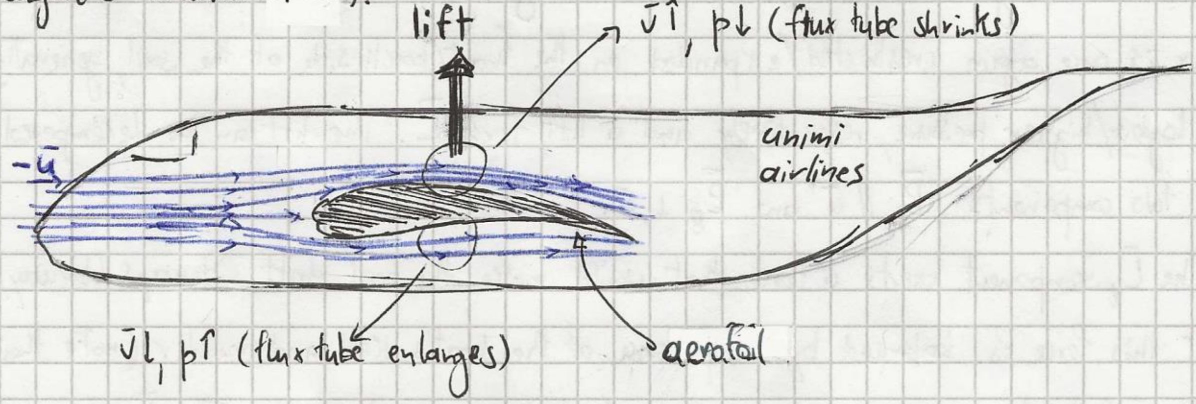
* = by the third law of motion, the force exerted by the spoon on the flow is directed upstream, i.e. it opposes the flow, hence a resistance.

Airplane flight

The mechanics of flight, in a nutshell, are not different from a spoon in the water jet out of a faucet; the latter with its concavity (designed to pick up a soup... but source of other forms of fluid dynamics fun), the former with its properly shaped wing (the AEROFOIL or AIRFOIL is the carefully designed cross sectional shape of an aircraft wing).

Let us have an airplane travelling at constant horizontal velocity $\vec{u}$; in a reference frame moving with the aircraft (=> airplane standing still) we get an air flow directed towards the front of the airplane and its wings. Due to the characteristic profile of the aerofoil (see figure), the flux tube is choked above / enlarged below the wing; the $p_{up} < p_{down}$ which means there is a vertical, upward-directed lift keeping the aircraft up in the sky (together with a horizontal drag slowing the plane, counteracted by the engine's forward thrust).



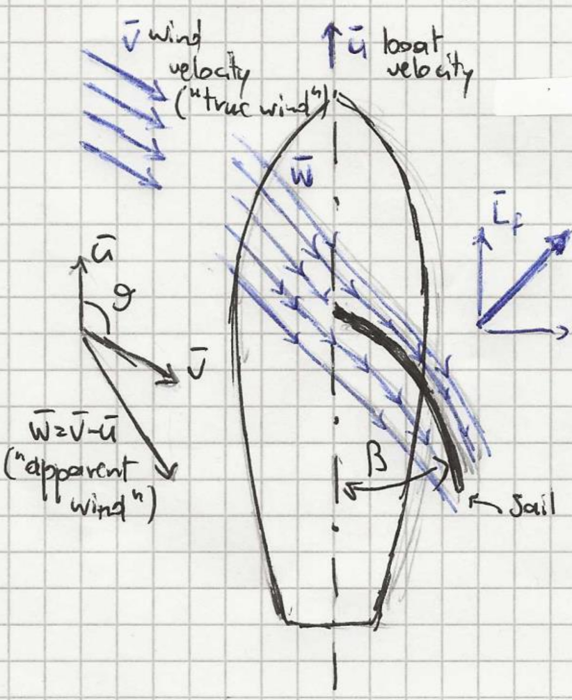
Sail boats and close-hauled sailing (sailing "into the wind")

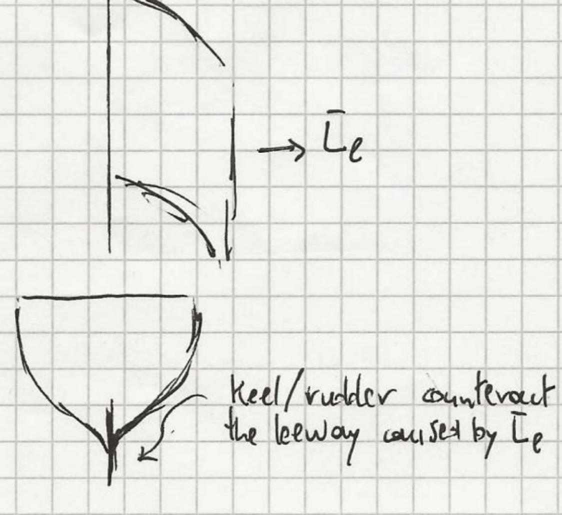
Sail boats can move in a direction almost facing an incoming wind current – not exactly, but at a relatively narrow angle (point of sail) with respect to the wind (close-hauled point of sail). This is made possible by the same effect experienced by an aerofoil.

Let us call $\bar{v}$ the wind velocity and $\bar{u}$ the observed (while not yet understood) boat velocity; hence $\bar{w} = \bar{v} - \bar{u}$ is the wind velocity with respect to the boat, i.e. the boat experiences a flux of air with velocity $\bar{w}$ and the sail must be rotated (hauled) with respect to the sailing direction (close-hauled) in such a way that it aligns more or less along $\bar{w}$ (while if the boat sails with the help of a wind from its back, i.e. it is "running downward", the sails should be orthogonal to $\bar{u}$).

In this configuration the wind flux runs along the two surfaces of the sail and the flux is once again constricted/expanded on the front/back side of the sail, generating a lower/higher pressure, respectively and a lift force $\bar{L}$. The lift can be decomposed in two components $\bar{L}_f \parallel \bar{u}$ and $\bar{L}_\ell \perp \bar{u}$.

The $\bar{L}_\ell$ component exerts a force that would make the boat drift sideways (leeway), but this force is balanced by the presence of the boat's keel and rudder (note that with a flat bottom and no rudder, sailing into the wind is not possible. The $\bar{L}_f$ component pushes the boat forward, along its forward direction $\bar{u}$ instead.





In rough terms, one would expect $\bar{L}_f$ to be a monotonically increasing function of $\bar{w}$, or, better, if we consider Bernoulli's equation (where Δp is related to $\Delta(v^2)$), an increasing function of w^2. If we call ϑ the angle between $\bar{u}$ and $\bar{v}$, $w^2 = v^2 + u^2 - 2uv\cos\vartheta$, then

- if $\vartheta > \pi/2 \Rightarrow \cos\vartheta < 0 \Rightarrow w^2$ increases for increasing boat speed u and so does the forward force $\bar{L}_f$;
- if $\vartheta < \pi/2 \Rightarrow \cos\vartheta > 0 \Rightarrow w^2, \bar{L}_f$ decrease with increasing u.

Notice that $\vartheta < \pi/2$ means that the true wind $\bar{v}$ is coming from behind (at least partially), and here we discover that sailing into the wind ("beating", with close-hauled sails) allows sailors to go faster than when running downwind. When the wind comes exactly from behind ($\vartheta = 0$) there can be no lift effect $\bar{P}_f$ at all, as is the case of wind exactly from the front ($\vartheta = \pi$); so a $\vartheta = \pi$ angle or close is called "no-go" zone, but very narrow angles, and hence narrow sail angle β (close-hauled sail), are allowed (depending on a number of aerodynamic properties of boat and sail).