Incompressibility condition: A physical perspective

We should always talk about an INCOMPRESSIBLE FLOW rather than an incompressible fluid: we shall see in the following that this property, which we have expressed so far as $\operatorname{div} \bar{v} = \phi$, is related to the properties of motion, i.e. of the flow, and not to the intrinsic features of the fluid.

Let us consider ρ as a function of a pair of thermodynamic quantities, say $\rho = \rho(p, s)$;

$\Rightarrow d\rho(p,s) = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds$ but $ds = \phi$ for ideal fluid (adiabatic flow)

$\Rightarrow d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp$; the study of acoustics would also tell us that

$\left(\frac{\partial p}{\partial \rho}\right)_s = c^2 \Rightarrow d\rho = \frac{1}{c^2} dp$ with c = speed of sound in the fluid under consideration;

since incompressibility means negligible variation of density, $d\rho/\rho \ll 1$,

the condition reads $\frac{d\rho}{\rho} = \boxed{\frac{1}{c^2} \frac{dp}{\rho} \ll 1}$. Let us elaborate this requirement.

Steady flow

Ignoring external force fields with potential u, Bernoulli's equation requires

$\frac{1}{2}\Delta(v^2) + \frac{\Delta p}{\rho} = \phi \Rightarrow \frac{\Delta p}{\rho} \sim \frac{1}{2} v^2$ as an order-of-magnitude estimate (if we think $\Delta(v^2)$ is the difference between v^2 in a generic, representative point in the flow with respect to a stagnation point where $v = \phi$ and $p = p_{max}$) $\Rightarrow$ incompressibility means

$\frac{1}{c^2}\frac{\Delta p}{\rho} \sim \frac{1}{2}\frac{v^2}{c^2} \ll 1 \Rightarrow \boxed{v \ll c}$, i.e. the velocity in the flow is much smaller than the velocity of propagation of sound waves.

Unsteady flow

CASE (A): The system does not feature an intrinsic typical time scale, i.e. a scale relevant to the description of the phenomena we observe (for instance the period of an oscillatory motion / a wave, a decay time scale, ...). Then the condition $\boxed{v \ll c}$ is enough.

Indeed, said l and u typical length and velocity scales in the flow, and since in Euler's equation we can say, in principle, that the terms $\partial\bar{v}/\partial t$ and $(\bar{v}\cdot\operatorname{grad})\bar{v}$ are comparable, we can write, in terms of order-of-magnitude estimates,

$\frac{\partial \bar{v}}{\partial t} \sim \frac{1}{\ell/u} u \Rightarrow \sim \frac{u^2}{\ell}$; $(\bar{v}\cdot \text{grad})\bar{v} \sim u \cdot \frac{1}{\ell} u \sim \frac{u^2}{\ell}$ as well ;

hence the right-hand side must scale accordingly, $-\frac{1}{\rho}\,\text{grad}\,p \sim \frac{u^2}{\ell}$;

but $\text{grad}\,p \sim \frac{dp}{\ell} \Rightarrow$: $\frac{u^2}{\ell} \sim \frac{1}{\rho}\frac{\Delta p}{\ell} \Rightarrow \Delta p \sim \rho u^2$

$\Rightarrow \frac{d\rho}{\rho} = \frac{1}{c^2}\frac{dp}{\rho} \sim \frac{1}{c^2}\frac{1}{\rho}\rho u^2 \sim \frac{u^2}{\ell^2} \ll 1 \Rightarrow \boxed{v \sim u \ll c}$ as found for steady flow.

CASE (3): If a typical time scale τ exists,

$\frac{\partial \bar{v}}{\partial t} \sim \frac{u}{\tau}$ while $(\bar{v}\cdot\text{grad})\bar{v} \sim u^2/\ell$

⊙ If $(\bar{v}\cdot\text{grad})\bar{v}$ prevails $\frac{u}{\tau} \ll \frac{u^2}{\ell}$, i.e. $\tau \gg \ell/u$, intrinsic time scale is very large and flow is almost a steady one

$\Rightarrow$ we are back to the condition $\boxed{v \ll c}$.

⊙ If the two terms are comparable or $\partial\bar{v}/\partial t$ dominates, Euler's eq. is, in order-of-magnitude estimate, $\frac{u}{\tau} \sim \frac{1}{\rho}\frac{\Delta p}{\ell} \Rightarrow \frac{\Delta \rho}{\rho} = \frac{1}{c^2}\frac{\Delta p}{\rho} \sim \frac{\ell u}{c^2 \tau}$

From the continuity equation $\frac{1}{\rho}\frac{D\rho}{Dt} = -\text{div}\,\bar{v}$, incompressibility requires negligible variation of ρ in time (along the flow) $\Rightarrow$

$\frac{1}{\rho}\frac{D\rho}{Dt} \ll \text{div}\,\bar{v}$; dimensionally $\frac{1}{\rho}\frac{\Delta\rho}{\tau} \ll \frac{u}{\ell}$

Combining with $\frac{\Delta\rho}{\rho} \sim \frac{\ell u}{c^2\tau} \Rightarrow \frac{\ell u}{c^2\tau^2} \ll \frac{u}{\ell} \Rightarrow \tau^2 \gg \ell^2/c^2 \Rightarrow \boxed{\tau \gg \ell/c}$

(or $c \gg \ell/\tau$)

which means that the typical time scale of changes in the flow is much larger than the propagation time of sound waves across the fluid domain (i.e. "instantaneous" propagation; indeed $c^2 = \partial p/\partial\rho$ and if ρ = constant $\Delta\rho \to \phi$ and $c \to \infty$).

Therefore when the flow features an intrinsic time scale, two conditions must be met to describe the flow as incompressible:

$$\boxed{\begin{array}{l} v \ll c \\ \tau \gg \ell/c \end{array}}$$

Note: A flow with $v \ll c$ is called <u>subsonic</u>. One might think air is definitely compressible, yet for subsonic flow air behaves essentially as incompressible.

Kelvin's circulation theorem - Conservation of circulation

We saw that $\frac{D}{Dt}\int_{\gamma(t)} f(\bar{x},t)\,dS_i = \int_{\gamma(t)}\left(f(\bar{x},t)\frac{\partial v_i}{\partial x_j} + \frac{D}{Dt}f(\bar{x},t)\delta_{ij}\right)dS_j$ along a curve $\gamma(t)$ in the fluid

and if $f = f_i$ i-th component of a vector function $\bar{f}(\bar{x},t)$

$$\frac{D}{Dt}\int_{\gamma(t)}\bar{f}\cdot d\bar{x} = \int_{\gamma(t)}\bar{f}\cdot d\bar{v} + \int_{\gamma(t)}\frac{D\bar{f}}{Dt}\cdot d\bar{x}$$

; with $\bar{f}=\bar{v}$ velocity of an ideal fluid ($\gamma(t)$ set of fluid particles in motion)

$$\frac{D}{Dt}\int_{\gamma(t)}\bar{v}\cdot d\bar{x} = \int_{\gamma(t)}\bar{v}\cdot d\bar{v} + \int_{\gamma(t)}\frac{D\bar{v}}{Dt}\cdot d\bar{x} = \int_{\gamma(t)} d\left(\tfrac{1}{2}v^2\right) + \int_{\gamma(t)}\frac{D\bar{v}}{Dt}\cdot d\bar{x}$$

If $\gamma(t)$ is a closed curve, we can define $\Gamma(t) = \oint_{\gamma(t)}\bar{v}\cdot d\bar{l}$ CIRCULATION and

the first integral (integral of an exact differential, depending only on the endpoints' values) $= \phi$

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_{\gamma(t)}\bar{v}\cdot d\bar{l} = \oint_{\gamma(t)}\frac{D\bar{v}}{Dt}\cdot d\bar{l}$$

Euler's equation states $\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\operatorname{grad}p - \operatorname{grad}u$;

if we can make the assumption that $\frac{1}{\rho}\operatorname{grad}p$ has a potential ψ, i.e. $\operatorname{grad}\psi = \frac{1}{\rho}\operatorname{grad}p$, we can take this calculation further. Landau's assumption is that of an isentropic fluid, such that $\psi = w$ specific enthalpy; it follows that

$$\frac{D\bar{v}}{Dt} = -\operatorname{grad}(w+u) \Rightarrow \frac{D\Gamma}{Dt} = \oint_{\gamma(t)}\frac{D\bar{v}}{Dt}\cdot d\bar{l} = -\oint_{\gamma(t)}\operatorname{grad}(w+u)\cdot d\bar{l} = -\int_{S(t)}\operatorname{curl}[\operatorname{grad}(w+u)]\cdot d\bar{S}$$

by Stokes theorem where $\gamma = \partial S$

and since $\operatorname{curl}(\operatorname{grad} f) = \phi \quad \forall f$

$$\boxed{\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_{\gamma(t)}\bar{v}\cdot d\bar{l} = \phi}$$

circulation is conserved along the motion of an isentropic ideal fluid (Kelvin's circulation theorem)

Important notes:

- As hinted at above, the request for isentropic flow is not strict; it is enough that $\exists\,\psi /$ $\operatorname{grad}\psi = \frac{1}{\rho}\operatorname{grad}p$. Notice that possibilities for this condition to be satisfied are
  - isentropic flow, as $dw = \cancel{T ds} + \frac{1}{\rho}dp = \frac{1}{\rho}dp \Rightarrow \psi = w$ enthalpy
  - isothermal flow, as $d\phi = \cancel{-s\,dT} + \frac{1}{\rho}dp = \frac{1}{\rho}dp \Rightarrow \psi = \phi$ Gibbs free energy
  - incompressible flow, as $\frac{1}{\rho}\operatorname{grad}p = \operatorname{grad}\left(\frac{p}{\rho}\right) \Rightarrow \psi = p/\rho$

⊙ Let us revise the calculation,

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_{\gamma(t)} \bar{v}\cdot d\bar{l} = \oint \frac{D\bar{v}}{Dt}\cdot d\bar{l} = -\oint_{\gamma(t)}\left(\frac{1}{\rho}\,\text{grad}\,p + \text{grad}\,u\right)\cdot d\bar{l} \underset{\text{by Stokes}}{=} -\int_S \text{curl}\left(\frac{1}{\rho}\,\text{grad}\,p + \text{grad}\,u\right)\cdot\hat{n}\,dS =$$

curl (grad u) = ∅

$$= -\int_S \text{curl}\left(\frac{1}{\rho}\,\text{grad}\,p\right)\cdot\hat{n}\,dS = -\iint_S \left[\frac{1}{\rho}\underbrace{\text{curl}(\text{grad}\,p)}_{\emptyset} + \underbrace{\text{grad}\left(\frac{1}{\rho}\right)}_{-\frac{1}{\rho^2}\text{grad}\,\rho}\times\text{grad}\,p\right]\cdot\hat{n}\,dS =$$

$$= \int_S \frac{1}{\rho^2}(\text{grad}\,\rho\times\text{grad}\,p)\cdot\hat{n}\,dS$$

and here we realize that $\text{grad}\,\rho\times\text{grad}\,p = \emptyset$ $\forall$ <u>barotropic fluid</u>

$$\Rightarrow \frac{d\Gamma}{dt} = \frac{D}{Dt}\oint_{\gamma(t)} \bar{v}\cdot d\bar{l} = \emptyset$$

Kelvin's circulation theorem holds for ideal, barotropic fluids (i.e. fluids where surfaces of uniform pressure and uniform density coincide, or equivalently $\rho = f(p)$ only).

⊙ If we shrink γ so much that it encloses an infinitesimal area, that is to say, a single streamline (in steady flow) / pathline (unsteady flow), then

$$\oint_{\gamma=\partial S} \bar{v}\cdot d\bar{l} = \int_S \text{curl}\,\bar{v}\cdot d\bar{a} \simeq \text{curl}\,\bar{v}\cdot\bar{S} = \text{constant}$$ due to Kelvin's theorem;

hence <u>vorticity $\bar{\omega} = \text{curl}\,\bar{v}$ is conserved along the flow.</u>

Potential flow

Conservation of circulation $\Gamma = \oint_{\gamma(t)} \bar{v}\cdot d\bar{l}$ induces observations of remarkable importance. For a steady flow, if there is a point where $\text{curl}\,\bar{v} = \phi$, if we consider the streamline passing through such point and a closed curve γ enclosing just this streamline, we can conclude that $\text{curl}\,\bar{v} = \phi$ along the whole streamline. If the flow is unsteady, this consideration is true for a pathline instead of a streamline. Remember that the flow must be isentropic (if collisions or turbulence occur, the whole thing breaks down).

Let us open up the scope and consider a steady flow that is irrotational ($\bar{\omega} = \text{curl}\,\bar{v} = \phi$) at the boundaries (e.g., uniform flow at infinity, in an infinite domain); then $\text{curl}\,\bar{v} = \phi$ on all streamlines, that is, in the whole domain. Or, let us say that the flow is unsteady but $\text{curl}\,\bar{v} = \phi$ everywhere at a certain time instant; then the flow must remain such for all subsequent times. In both cases we have a POTENTIAL or IRROTATIONAL FLOW, i.e. such that $\text{curl}\,\bar{v} = \phi$ everywhere; this also implies that $\bar{v}$ can be expressed as the gradient of a scalar field, a potential, φ: $\underline{\underline{\bar{v} = \text{grad}\,\varphi}}$ ($\text{rot}(\text{grad}\,f) = \phi \quad \forall f$). Thus we say φ is named VELOCITY SCALAR POTENTIAL. In a potential flow there are no closed streamlines ($\oint_\gamma \bar{v}\cdot d\bar{l} = \phi$).

Recall that Euler's equation, when $\frac{1}{\rho}\,\text{grad} = \text{grad}\,\psi$, can be written as

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\text{grad})\,\bar{v} = -\,\text{grad}(\psi + u) \quad \text{and also reworked as}$$

$$\frac{\partial \bar{v}}{\partial t} + \text{grad}\left(\frac{1}{2}v^2\right) - \bar{v}\times\text{curl}\,\bar{v} = -\,\text{grad}(\psi + u) \quad ; \text{ with } \text{curl}\,\bar{v} = \phi, \quad \bar{v} = \text{grad}\,\varphi,$$

$$\frac{\partial}{\partial t}(\text{grad}\,\varphi) + \text{grad}\left(\frac{1}{2}v^2 + \varphi + u\right) = \phi \quad \Rightarrow \quad \text{grad}\left(\frac{\partial \varphi}{\partial t} + \frac{1}{2}v^2 + \psi + u\right) = \phi$$

that is to say, $\frac{\partial \varphi}{\partial t} + \frac{1}{2}v^2 + \psi + u = f(t)$ arbitrary function of time only

and $f(t)$ can be set $= \phi$ because φ is not uniquely defined by its definition $\bar{v} = \text{grad}\,\varphi$; in other words, we can always operate a gauge transformation

$\varphi' = \varphi - \int f(t)\,dt$ so that

$\text{grad}\,\varphi' = \text{grad}\,\varphi = \bar{v}$ anyway, and using φ' in Euler's equation through the whole calculation made above,

$$\frac{\partial \varphi'}{\partial t} + \frac{1}{2}v^2 + \psi + u = \frac{\partial \varphi}{\partial t} + \frac{1}{2}v^2 + \psi + u - f(t) = f(t) - f(t) = \phi$$

$$v^2 = |\text{grad}\,\varphi'|^2 = |\text{grad}\,\varphi|^2$$

So we can be reassured that it is perfectly legitimate to write

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}v^2 + \psi + u = \frac{\partial \varphi}{\partial t} + \frac{1}{2}|\text{grad}\,\varphi|^2 + \psi + u = \phi$$

Notes:

- For a steady flow, $\partial\varphi/\partial t = \phi \Rightarrow \boxed{\frac{1}{2}v^2 + \psi + u = \phi}$ or $\boxed{\frac{1}{2}|\text{grad}\,\varphi|^2 + \psi + u = \phi}$

generalized Bernoulli's equation

There is a significant difference with respect to the Bernoulli's equation seen before to hold for any steady flow of ideal fluids: In the latter case, the result is the conservation of a quantity on an individual streamline (a different constant is found for each streamline); for potential flow, the constant quantity is the same (zero) everywhere.

- For incompressible flow, $\psi = p/\rho$: ε is conserved separately along the flow and we can ignore it in the equation, $\Rightarrow \boxed{\frac{1}{2}v^2 + p/\rho + u = \phi}$ everywhere.

- Incompressibility is equivalent to $\text{div}\,\bar{v} = \phi$;

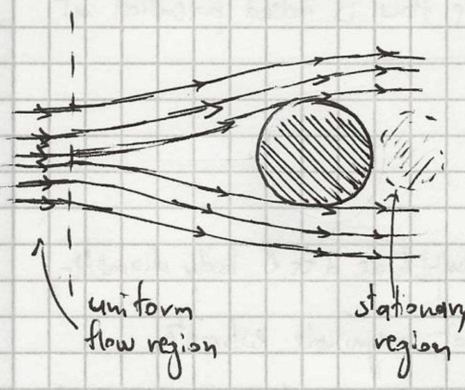
  so $\text{div}\,\bar{v} = \text{div}(\text{grad}\,\varphi) = \boxed{\nabla^2\varphi = \phi}$ a Laplace equation!

So for incompressible potential flow, instead of Euler's equation, we can get the dynamics (velocity field) just by using a (simpler) Laplace equation with its boundary conditions (b.c.). These follow from Euler's eq. own b.c. and potential flow conditions: If the domain is infinite, we shall ask for uniform velocity at infinity; on any solid surface such as walls or obstacles, like immersed bodies, $v_n|_{body}$ relative normal component of the fluid velocity with respect to the body must be zero (the fluid cannot penetrate the solid object):

$$\begin{cases} \text{grad}\,\varphi = \bar{v}_\infty \text{ uniform at infinity} \\ v_n|_{body} = \text{grad}\,\varphi \cdot \hat{n}|_{body} = \begin{cases} \phi & \text{for fixed bodies} \\ v_n(t) & \text{of the body itself} \end{cases} \end{cases}$$

Since an ideal fluid exerts no friction, no condition is asked for the parallel component of $\bar{v}$ with respect to the body at its surface; the fluid can slip along the body freely (FREE-SLIP CONDITION).

⊙ We should ask ourselves whether the description of potential flow is physically reasonable, or at least when/when not. Landau discusses this point at length, considering the case of <u>flow past an obstacle.</u>



Consider an ideal fluid flow starting very far on the left (at infinity, in a fair approximation) with uniform $\vec{v}$. We expect to be able to treat this flow as potential since $\mathrm{curl}\,\vec{v} = \phi$ there and then along the whole flow. Hence circulation Γ on a closed loop γ around any streamline is zero.

But if there is an obstacle somewhere in the flow, some streamlines must exist that graze the surface of such solid body. No closed loop γ can exist in the fluid around a streamline in that region, as γ will intersect the body. Streamlines following the body's contour for a while must detach from it at some point, and continue into the fluid. This means there are solutions to the equations of motion of the fluid that admit <u>separation</u> and, in turn, surfaces of <u>tangential discontinuity</u> i.e. where the fluid velocity will experience a discontinuity or, in other words, fluid layers slide one with respect to another. With reference to the figure, in the region past the body there is moving fluid on the sides, and a zone of stationary fluid right behind the obstacle. If there is a velocity gradient normal to the velocity itself (shear: fluid layers are sliding one onto another), $\mathrm{curl}\,\vec{v} \neq \phi$.

If such discontinuous flow possibility is accepted, the solution to the equations of motion is not unique: There exist infinite solutions where tangential discontinuities occur. But these discontinuities are unstable and lead to <u>turbulence</u> (chaotic motion), breaking the isentropic hypothesis and hence the possibility of treating the flow as a potential flow.

Is everything lost? Not entirely. The solution is determined by the fact that the ideal fluid is an approximation; it does not really exist, as any fluid possesses a certain viscosity; while well within the fluid flow viscosity may be negligible, in the vicinity of a solid obstacle (and together with the surface properties of the obstacle itself) it must be considered and will actually play a key role in determining the features of the flow around the body. Viscosity leads to the formation of a thin layer of fluid on the body surface (called <u>boundary layer</u>), and a wake behind the body. The properties of such boundary layer determine the specific physical solution to the problem of the flow — in general, a solution where separation and turbulence occur but in some instances (e.g., when the shape of the obstacle is simple or streamlined) it may happen that the flow is indeed potential at least when far from the boundary layer and wake.

⊙ Immersed body performing low-amplitude oscillations

We can show that if a body in the fluid oscillates with amplitude $a \ll l$ body diameter, the flow can be treated as irrotational, making an order-of-magnitude estimate.

Let us consider Euler's eq. $\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \text{grad})\, \bar{v} = -\text{grad}(\psi + u)$ $\quad (\psi \,/\, \text{grad}\,\psi = \frac{1}{\rho}\,\text{grad}\,p)$

Fluid velocity $\bar{v}$ varies of an amount $\sim \bar{u}$ (body velocity) over distances $\sim l \Rightarrow$

$\frac{\partial v_i}{\partial x_j} \sim \frac{u}{l}$ and $\bar{v}$ itself being $\sim \bar{u}$ implies $(\bar{v} \cdot \text{grad})\bar{v} \sim u^2/l$

If oscillation frequency $\omega = u/a \Rightarrow \frac{\partial \bar{v}}{\partial t} \sim \omega u = u^2/a$

Then the left-hand side of Euler's equation is the sum of two terms

$\sim \frac{u^2}{a} + \frac{u^2}{l}$ with $a \ll l \Rightarrow$ the second term $(\bar{v} \cdot \text{grad}\, \bar{v})$ is negligible $\Rightarrow$

Euler's eq. is approximately $\frac{\partial \bar{v}}{\partial t} = -\text{grad}(\psi + u)$ by neglecting the advective term;

applying the curl to this equation,

$$\frac{\partial}{\partial t}(\text{curl}\,\bar{v}) = -\text{curl}\left[\text{grad}(\psi + u)\right] = \phi \qquad \text{since} \quad \text{curl}(\text{grad}\, f) = \phi \quad \forall f$$

$$\Rightarrow \frac{\partial}{\partial t}(\text{curl}\,\bar{v}) = \phi \quad \Rightarrow \quad \text{curl}\,\bar{v} = \text{constant}$$

but since in an oscillatory motion the average velocity is zero, this constant is also zero and we demonstrate here that $\underline{\text{curl}\,\bar{v} = \phi}$, i.e. we have potential flow.

Drag in potential flow past a body

We shall investigate in detail what happens if a solid body is immersed within an ideal fluid in a state of potential flow – or equivalently, the fluid motion induced when a solid body is set in motion within the initially stationary fluid; the former case is derived from the latter by switching to a coordinate system sitting on the body.

Let us indeed consider the second case, where the body moves in an initially stationary fluid. If we assume a very large (infinite) domain, the problem can be stated as

$$\begin{cases} \nabla^2 \varphi = 0 \\ \bar{v}(\bar{r} \to \infty) = 0 \quad (\mathrm{grad}\,\varphi(\bar{r}\to\infty) = 0) \quad \text{fluid at rest very far from the perturbing body} \\ v_n(\text{fluid}) = v_n(\text{body}) \quad \text{free-slip condition on the body surface} \end{cases}$$

At infinity $\mathrm{grad}\,\varphi = 0 \Rightarrow \varphi = \text{uniform}$ (let us say $= 0$).

Consider the origin of the cs to be inside the body at a certain time instant t; possible solution for a Laplace equation are $1/r$, $\mathrm{grad}(1/r)$ and higher-order derivatives of $1/r$; all of these vanish at $r \to \infty$; the asymptotic solution, at great distance r is therefore

$$\varphi = -\frac{a}{r} + \bar{A}\cdot \mathrm{grad}\,\frac{1}{r} + \ldots \text{ (higher-derivative terms) with } \bar{A} \text{ constant vector}$$

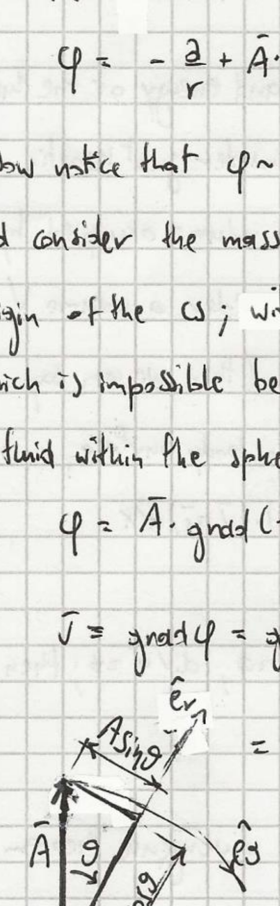
Now notice that $\varphi \sim a/r$ yields $\bar{v} = -\mathrm{grad}(a/r) = a\bar{r}/r^3$

and consider the mass flow rate through a spherical surface of radius R centered in the origin of the cs; with $v_r(R) = a/R^2$, $\quad q = \rho \frac{a}{R^2} 4\pi R^2 = 4\pi\rho a \neq 0$

which is impossible because the surface is closed, the flow is incompressible and there is no source of fluid within the sphere $\Rightarrow$

$$\varphi = \bar{A}\cdot \mathrm{grad}(1/r) = -\bar{A}\cdot \hat{e}_r / r^2 \quad \text{and}$$

$$\bar{v} = \mathrm{grad}\,\varphi = \mathrm{grad}\left(-\bar{A}\cdot\hat{e}_r/r^2\right) = \mathrm{grad}\left(-A\cos\vartheta/r^2\right) = \hat{e}_r \frac{\partial}{\partial r}\left(-\frac{A\cos\vartheta}{r^2}\right) + \frac{\hat{e}_\vartheta}{r}\frac{\partial}{\partial\vartheta}\left(-\frac{A\cos\vartheta}{r^2}\right) =$$



$$= \frac{2A\cos\vartheta}{r^3}\hat{e}_r + \frac{A\sin\vartheta}{r^3}\hat{e}_\vartheta = \frac{3A\cos\vartheta}{r^3}\hat{e}_r - \frac{1}{r^3}\underbrace{\left(A\cos\vartheta\,\hat{e}_r - A\sin\vartheta\,\hat{e}_\vartheta\right)}_{\bar{A}}$$

$$\Rightarrow \bar{v}(\bar{r}) = \frac{3(\bar{A}\cdot\hat{e}_r)\hat{e}_r - \bar{A}}{r^3}$$

This vector $\bar{A}$ can be determined once we know and use explicitly geometry and velocity of the body, that is, in mathematical terms, when we take into account the boundary conditions at the surface of the moving body (zero relative normal fluid velocity at the surface) and solve completely the Laplace equation $\forall \bar{r}$.

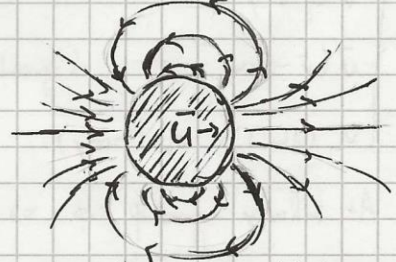
Example: A spherical body of radius R moves with steady velocity $\bar{u}$.

The b.c. on the sphere's surface is $\bar{v}(R)\cdot\hat{e}_r = \bar{u}\cdot\hat{e}_r$; using the form obtained for $\bar{v}(\bar{r})$ and plugging in the b.c.,

$$\bar{v}(R)\cdot\hat{e}_r = v_r(R) = \left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = -\bar{A}\cdot\hat{e}_r \left.\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right|_{r=R} = \bar{A}\cdot\hat{e}_r\,\frac{2}{R^3} = \bar{u}\cdot\hat{e}_r$$

$$\Rightarrow \bar{A} = \frac{1}{2}R^3\bar{u} \quad \Rightarrow \quad \bar{v} = \frac{R^3}{2r^3}\left[3(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}\right]$$

and $\varphi = -\bar{A}\cdot\hat{e}_r/r^2 = -\frac{R^3}{2}\frac{\bar{u}\cdot\hat{e}_r}{r^2} = -\frac{R^3}{2}\frac{\bar{u}\cdot\bar{r}}{r^2}$



velocity field qualitative map

Notice that φ looks like the potential of an electric dipole (which gives rise to an irrotational electric field, indeed). Also remember the solution is asymptotical, i.e. it is not really correct in the close proximity of the sphere.

In physical terms, $\bar{A}$ is related to quantities like the momentum and energy of the fluid (since we deal with an incompressible flow, ε is conserved and we can consider just kinetic energy). With an integration over the whole volume (excluding the volume occupied by the sphere), the total kinetic energy is $E_K = \int \rho v^2 d^3x$. Let us consider a volume V as a spherical region centered in the origin of the cs and radius R (then we can say $R\to\infty$); then, if $\bar{u}$ = body velocity, V_0 = body volume, $S_0 = \partial V_0$ body surface,

$$\int_{V-V_0} v^2 d^3x = \int_{V-V_0} u^2 d^3x + \int_{V-V_0} (\bar{v}+\bar{u})\cdot(\bar{v}-\bar{u})\,d^3x = u^2(V-V_0) + \int_{V-V_0}(\bar{v}+\bar{u})\cdot(\bar{v}-\bar{u})\,d^3x$$

Let us rewrite $\bar{v}+\bar{u} = \mathrm{grad}(\varphi+\bar{u}\cdot\bar{r})$ and recall that $\mathrm{div}\,\bar{v} = 0$, $\mathrm{div}\,\bar{u} = 0$; then

$$\int_{V-V_0} v^2 d^3x = u^2(V-V_0) + \int_{V-V_0}\mathrm{grad}(\varphi+\bar{u}\cdot\bar{r})\cdot(\bar{v}-\bar{u})\,d^3x =$$

$$= u^2(V-V_0) + \int_{V-V_0}\mathrm{div}\left[(\varphi+\bar{u}\cdot\bar{r})(\bar{v}-\bar{u})\right]d^3x = \text{using the divergence theorem}$$

$$\left(\mathrm{div}\left[(\varphi+\bar{u}\cdot\bar{r})(\bar{v}-\bar{u})\right] = \mathrm{grad}(\varphi+\bar{u}\cdot\bar{r})\cdot(\bar{v}-\bar{u}) + (\varphi+\bar{u}\cdot\bar{r})\,\underbrace{\mathrm{div}(\bar{v}-\bar{u})}_{0}\right)$$

$$= u^2(V-V_0) + \int_{S+S_0} (\varphi + \bar{u}\cdot\bar{r})(\bar{v}-\bar{u})\cdot d\bar{S} \qquad (S = \partial V,\ S_0 = \partial V_0)$$

Since $v_n = u_n$ on the body's surface, the part of the flux integral over S_0 is zero. If now we make $R \to \infty$ and use the expressions for φ and $\bar{v}$,

$$\varphi(\bar{r}) = -\bar{A}\cdot\hat{e}_r/r^2, \qquad \bar{v}(\bar{r}) = [3(\bar{A}\cdot\hat{e}_r)\hat{e}_r - \bar{A}]/r^3, \quad \text{evaluated in } r=R,$$

$$\int_V v^2 d^3x = u^2(V-V_0) + \int_S \left[-\frac{\bar{A}\cdot\hat{e}_r}{R^2} + \bar{u}\cdot R\hat{e}_r\right]\left[\frac{3(\bar{A}\cdot\hat{e}_r)\hat{e}_r - \bar{A}}{R^3} - \bar{u}\right]\cdot R^2 d\Omega\, \hat{e}_r =$$

($d\bar{S} = dS\hat{e}_r = R^2 d\Omega \hat{e}_r$ with $d\Omega$ solid angle)

neglecting the terms that vanish for $R\to\infty$,

$$= u^2(V-V_0) + \int_{S_\infty} \left[(\bar{A}\cdot\hat{e}_r)\bar{u} + 3(\bar{A}\cdot\hat{e}_r)(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{A}(\bar{u}\cdot\hat{e}_r) - R^3(\bar{u}\cdot\hat{e}_r)\bar{u}\right]\cdot\hat{e}_r\, d\Omega =$$

$$= u^2(V-V_0) + \int_{S_\infty} \left[3(\bar{A}\cdot\hat{e}_r)(\bar{u}\cdot\hat{e}_r) - (\bar{u}\cdot\hat{e}_r)(\bar{u}\cdot\hat{e}_r)R^3\right] d\Omega$$

The integration over the solid angle is equivalent to averaging the integrand over all direction the unit vector $\hat{e}_r$ can take and multiplying over the full solid angle 4π. We have to average expressions taking a form of the type $(\bar{A}\cdot\hat{e}_r)(\bar{B}\cdot\hat{e}_r) = A_i e_{r_i} B_k e_{r_k}$, with $\bar{A}, \bar{B}$ constant vectors. We can calculate

$$\langle(\bar{A}\cdot\hat{e}_r)(\bar{B}\cdot\hat{e}_r)\rangle = A_i B_k \langle e_{r_i} e_{r_k}\rangle = \frac{1}{3}\delta_{ik} A_i B_k = \frac{1}{3}A_i B_i = \frac{1}{3}\bar{A}\cdot\bar{B}$$

one can show that $\langle e_{r_i} e_{r_k}\rangle = \frac{1}{3}\delta_{ik}$

Using this result and the fact that V is a sphere, so $V = \frac{4\pi}{3}R^3$,

$$\int_{V_\infty - V_0} v^2 d^3x = \frac{4\pi}{3}R^3 u^2 - V_0 u^2 + 4\pi\cdot\frac{1}{3}3(\bar{A}\cdot\bar{u}) - 4\pi\cdot\frac{1}{3}R^3 u^2 = 4\pi(\bar{A}\cdot\bar{u}) - V_0 u^2$$

$$\Rightarrow E_K = \frac{1}{2}\rho\int_{V_\infty - V_0} v^2 d^3x = \frac{1}{2}\rho\left[4\pi(\bar{A}\cdot\bar{u} - V_0 u^2)\right]$$

While the exact determination of $\bar{A}$ requires solving the Laplace equation for φ with its b.c., we can draw some qualitative observations about it if we consider that the Laplace equation is a linear eq. in φ with b.c. linear in φ and $\bar{u}$. Hence $\bar{A}$ must also be a linear function of $\bar{u}$ and the kinetic energy written above is a quadratic function of $\bar{u}$, which we can write in the form

$$E_K = \frac{1}{2} m_{ik} u_i u_k$$

where m_{ik} is a 2nd-order symmetrical tensor called INDUCED-MASS TENSOR (obtaining $\bar{A}$ means that we also get $\underline{\underline{m}}$).

If E_K is obtained, we can also derive the total momentum $\bar{P}$ of the fluid using the relationship $dE_K = \bar{u}\cdot d\bar{P}$ (valid only for an ideal incompressible flow, where work is completely converted into kinetic energy as internal energy is conserved) *. Therefore we get

$$\underline{P_i = m_{ik} u_k}$$

and since $E_K = \frac{1}{2}\rho\left(4\bar{u}\bar{A}\cdot\bar{u} - V_0 u^2\right) \Rightarrow \underline{\underline{\bar{P} = 4\bar{u}\rho\bar{A} - \rho V_0 \bar{u}}}$

The momentum per unit time transferred by the body to the fluid is $d\bar{P}/dt$; the reaction force $\bar{F}$ of the fluid on the body is clearly equal and opposite $\Rightarrow$ $\underline{\underline{\bar{F} = -d\bar{P}/dt}}$ and this force can be decomposed into

$\bar{F}_{\parallel} = (\bar{F}\cdot\bar{u}/u)\,\bar{u}/u$ DRAG FORCE, a resistance parallel to the body velocity $\bar{u}$

$\bar{F}_{\perp} = \bar{F} - \bar{F}_{\parallel} = \bar{F} - (\bar{F}\cdot\bar{u}/u)\,\bar{u}/u$ LIFT FORCE

An important note: If we had a potential flow past a solid body in uniform linear motion (i.e. with $\bar{u}$ = constant) within an ideal fluid, constant $\bar{u}$ would imply constant $\bar{P}$ $\Rightarrow$ $\bar{F} = \emptyset$, that is to say, no drag or lift and forces on the body perfectly balanced, which contradicts actual observation of drag. This contradiction is known as d'Alembert's paradox. On the other hand, the existence of a drag implies the application of a work from an external force for the uniform motion to be maintained, a work converted into kinetic energy or dissipation in the fluid, which would result in a continual flow of energy towards infinity in the fluid. Again, the definition of ideal fluid forbids the occurrence of dissipation, and the fluid velocity rapidly decreases for increasing distance from the body, preventing energy flow to infinity.

As we already mentioned, the potential flow approximation is nevertheless not possible in the whole fluid and the paradox is due to neglecting the viscous boundary layer around the body, where high velocity gradients occur, with generation of vorticity and viscous dissipation of

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* = Let us have an external force $\bar{F}^{ext}$ applied on the body, that is also subjected to the force from the fluid, equal and opposite to the force $d\bar{P}/dt$ exerted by the body on the fluid; hence for the body we say $\bar{F}^{ext} - d\bar{P}/dt = M\dot{\bar{u}}$ and with scalar multiplication by $\bar{u}\,dt$, $dW = \bar{F}^{ext}\cdot\bar{u}\,dt = M\dot{\bar{u}}\cdot\bar{u}\,dt + \bar{u}\cdot d\bar{P} =$
$= dE_{body} + \bar{u}\cdot d\bar{P}$; but dW is the only work on the system $= dE_{body} + dE_{fluid}$ (for incompressible ideal flow) $\Rightarrow$ $dE_{fluid} = \bar{u}\cdot d\bar{P}$

energy and flow separation (indeed the theory of boundary layers was developed by Ludwig Prandtl, in a way, to address this paradox).

Landau also remarks that the reasoning about the paradox is formulated for an infinite fluid domain; but if that is not the case, and for instance the body is moving along the free surface of the fluid, waves will be generated that propagate on the surface and carry energy to infinity, resulting in a resistance force on the body called <u>wave drag</u> (indeed ducks and cetaceans swim underwater to reduce the wake and energy dissipation (so that swimming is less tiring).

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With the discussion above we have finally set up quantities and equations necessary to solve a problem like the dynamics for a fluid system with an immersed body.

⊙ For instance, let us consider a body performing oscillations inside an ideal fluid, within a potential flow approximation, under the action of a force $\bar{F}_{ext}$. The force causes a variation of momentum of the body and, in turn, of the fluid; that is,

$$\underbrace{M\frac{d\bar{u}}{dt}}_{\text{body}} + \underbrace{\frac{d\bar{P}}{dt}}_{\text{fluid}} = \bar{F}_{ext} \quad ; \quad \text{with} \quad P_i = m_{ik} u_k ,$$

$$M\frac{du_i}{dt} + m_{ik}\frac{du_k}{dt} = f_i , \quad \text{i.e.} \quad \boxed{\frac{du_k}{dt}\left(M\delta_{ik} + m_{ik}\right) = f_i} \quad \text{equation of motion for the immersed body}$$

⊙ Conversely, let us imagine the fluid is set in oscillatory motion by an external force, thus transferring momentum to the immersed body. Again, the fluid is ideal and the flow is incompressible; the body is small with respect to to spatial scale of change of the fluid velocity (V_0 body volume). If we replaced the body with an equal volume of fluid, the unperturbed fluid velocity in such volume would be $\bar{v}$, the momentum of this fluid volume would be $\rho V_0 \bar{v}$ and the force acting on it would be $\rho V_0 \, d\bar{v}/dt$. Since the body is not completely carried around by the fluid, but rather dragged with it, it acquires a certain velocity $\bar{u} \neq \bar{v}$, i.e. a relative motion with respect to the fluid and transfers some momentum to the fluid $m_{ik}(u_k - v_k)$, or a force to the fluid $m_{ik}\frac{d}{dt}(u_k - v_k)$; symmetrically, the body experiences an equal and opposite force. The total force acting on the body finally reads

$$\rho V_0 \frac{dv_i}{dt} - m_{ik}\frac{d}{dt}(u_k - v_k) = \frac{d}{dt}(M u_i) \quad \text{and by time integration}$$

$$(M\delta_{ik} + m_{ik})\, a_k = (m_{ik} + \rho V_0 \delta_{ik}) v_k \quad + \text{ constant}$$

where the constant is zero, because $\bar{u} = \phi$ if $\bar{v} = \phi$. This equation yield $\bar{u}$ body velocity if $\bar{v}$ fluid velocity is known. Notice that if the body's density is equal to the fluid's,
$M = \rho V_0 \implies \bar{u} = \bar{v}$.

(*) As an example let us reprise the sphere moving with velocity $\bar{u}$ within a fluid that is at rest very far from the body (infinity). If we consider a spherical CS that at a given instant t is centered in the sphere's center, we know already that the fluid $\bar{v}$ reads

$$\bar{v}(\bar{r}) = \frac{R^3}{2r^3}\left[(3\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}\right] \qquad (\bar{A} = R^3\bar{u}/2)$$

so that the total momentum transfered to the fluid is

$$\bar{P} = 4\pi\rho\bar{A} - \rho V_0 \bar{u} = 2\pi\rho R^3 \bar{u} - \rho\frac{4\pi}{3}R^3\bar{u} = \frac{2\pi}{3}\rho R^3 \bar{u}$$

and since $P_i = m_{ik} u_k$, the induced mass tensor reads

$$m_{ik} = \frac{2\pi}{3}\rho R^3 \delta_{ik}$$

and the drag exerted by the fluid on the body is

$$\bar{F} = -\frac{d\bar{P}}{dt} = -\frac{2\pi}{3}\rho R^3 \frac{d\bar{u}}{dt}$$

so if $\bar{f}$ is the force acting on the body to keep it moving, and $M = \frac{4\pi}{3}\rho_0 R^3$ the mass of the body of density ρ_0, the equation of motion $\frac{du_k}{dt}(M\delta_{ik} + m_{ik}) = f_i$ becomes

$$\underbrace{\frac{4\pi}{3}R^3\left(\rho_0 + \frac{1}{2}\rho\right)}_{\doteq\, m}\frac{d\bar{u}}{dt} = \bar{f}$$

i.e. $m\frac{d\bar{u}}{dt} = \bar{f}$, where m is the VIRTUAL or EFFECTIVE MASS of the sphere: besides M, we must account for the fluid's inertia

(*) Conversely, let us consider the "mirror problem" where the fluid is moving and also sets the body in motion. The general equation

$$(M\delta_{ik} + m_{ik})\,a_k = (m_{ik} + \rho V_0 \delta_{ik}) v_k \rightarrow \text{becomes} \rightarrow \frac{4\pi}{3}R^3\left(\rho_0 + \frac{\rho}{2}\right)\bar{u} = \frac{4\pi}{3}R^3\left(\frac{\rho}{2} + \rho\right)\bar{v}$$

$$\implies \boxed{\bar{u} = \frac{3\rho}{2\rho_0 + \rho}\,\bar{v}}$$

and let us consider an oscillatory motion (potential flow approximation is ok): Notice that if $\rho_0 = \rho$ the body is perfectly carried around by the fluid; if $\rho_0 > \rho$ the sphere "lags behind" the fluid; if $\rho_0 < \rho$ the sphere "goes ahead".