A supposedly fun thing with tensors (I'll never do again)

We recall here some well-known concepts (scalars, vectors) and then expand to include tensors, revising all these objects based on some very specific properties they possess, i.e. their TRANSFORMATION LAWS. For this treatment we refer to the first two chapters of "Mathematics Applied to Continuum Mechanics", Lee. A. Segel (Dover Publications, Inc. 1987). Theorems will be numbered according to this book.

We stick to a description within the context of cartesian frames of reference, hence the name of cartesian vectors and tensors.

In very simple terms we know

- a <u>scalar</u> is defined purely by its <u>magnitude</u>, that is a real number
- a <u>vector</u> is defined by its <u>magnitude</u> and <u>direction</u>.

But in other (algebraic) terms

a vector (or better a <u>POLAR VECTOR</u>) is a quantity whose components transform according to the same transformation law of a <u>position vector</u> upon rotation of the coordinate axes.

Such rotation brings us from an original coordinate system (cs) with unit coordinate vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ to a new cs with unit coordinate vectors (ucv) $\hat{e}_1', \hat{e}_2', \hat{e}_3'$.

The vector $\vec{x}$ is thus identified by the components x_i $(i=1,2,3)$ in the original cs

→ and x_i' in the rotated cs.

If we call l_{ij} the cosine of the angle between $\hat{e}_i$ and $\hat{e}_j'$ (i-axis of original cs / j-axis of the rotated cs; by convention, the second index is the one referring to the primed cs), the transformation law is

$$x_j' = x_1 l_{1j} + x_2 l_{2j} + x_3 l_{3j} = \sum_{i=1}^{3} x_i l_{ij}$$ or, using the summation convention,

$$\boxed{x_j' = x_i l_{ij}}$$ → hence the inverse transformation $x_i = x_j' l_{ij}$

Note: The summation convention omits the $\sum$ summation symbol, with the understanding that for any index repeated twice in a terms of an expression (also called a <u>dummy index</u>), a summation is implied over all the range of values the index can take.

As a consequence, when using this convention no index can be written more than twice in any terms of an algebraic expression. An index appearing only once is called <u>free</u>.

Notice that $l_{ij} l_{kj} = \delta_{ik}$ (and similarly $l_{ji} l_{jk} = \delta_{ik}$),

where we make use of the <u>Kronecker delta</u> symbol $\delta_{ij} \begin{cases} = 1 & \text{if } i=j \\ = \phi & \text{if } i \neq j \end{cases}$

and the relationship holds since

$$x_j' = \sum_{i=1}^{3} x_i l_{ij} = \sum_{i=1}^{3} x_i \cos(\hat{e}_i, \hat{e}_j')$$

Now, x_i can be written using the inverse transformation $x_i = \sum_{j=1}^{3} x_j' \cos(\hat{e}_j', \hat{e}_i)$;

in order to avoid repetion of indices we insert x_j' in this expression in this way:

$$x_i = \sum_{j=1}^{3} \sum_{k=1}^{3} x_k \cos(\hat{e}_k, \hat{e}_j') \cos(\hat{e}_j', \hat{e}_i) = \sum_{k=1}^{3} x_k \underbrace{\sum_{j=1}^{3} \cos(\hat{e}_k, \hat{e}_j') \cos(\hat{e}_j', \hat{e}_i)}$$

This is the projection of the unit vector $\hat{e}_i$ upon the $\hat{e}_k$ direction, and by orthogonality, such projection is 1 when $i=k$ or ϕ when $i \neq k$; $\Rightarrow \delta_{ik}$

and indeed we called $\left. \begin{array}{l} l_{kj} = \cos(\hat{e}_k, \hat{e}_j') \\ l_{ij} = \cos(\hat{e}_j', \hat{e}_i) \end{array} \right\} \Rightarrow$ so $\underline{l_{ij} l_{kj} = \delta_{ij}}$ (q.e.d.)

Notice also that $l_{ji} = l_{ij}^T$ element of the transpose matrix, but

$l_{ji} l_{jk} = \delta_{ik} \Rightarrow l_{ij}^T l_{jk} = \delta_{ik}$; in standard notation we would write

$\sum_{j=1}^{3} l_{ij}^T l_{jk} = A_{ik}$ and notice that A_{ik} is the result of a matrix product; now by saying $A_{ik} = \delta_{ik}$ we are saying $\underline{\underline{A}} = \mathbb{1}$ identity matrix

$\Rightarrow \underline{\underline{l}}^T \underline{\underline{l}} = \mathbb{1}$ means $\underline{\underline{l}}^T = \underline{\underline{l}}^{-1}$, i.e. the transpose of $\underline{\underline{l}}$ is also its inverse matrix.

This should actually not come as too surprising; the readers may remember from their past dabbling with geometry that the rotation matrix belongs to the class of ORTHOGONAL TRANSFORMATIONS; such matrices $\underline{\underline{R}}$ describe ISOMETRIC transformations in $\mathbb{R}^n$ (i.e. transformation preserving the scalar product, that is angles and lengths, or, in other word, <u>metric-preserving</u>) such that

$\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$ and $\Rightarrow \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T \underline{\underline{R}} = \mathbb{1}$ and $\det \underline{\underline{R}} = \pm 1$

In particular, $\det \underline{\underline{R}} = 1 \iff$ PROPER ROTATION (direct isometry), transforms a right-handed CS in another right-handed CS

$\det \underline{\underline{R}} = -1 \iff$ IMPROPER ROTATION (indirect isometry), transforms a right-handed cs into a left-handed cs: in other words, it is a proper rotation **plus** at least one reflection with respect to one of the axes; for instance,

$$\underline{R} = \begin{pmatrix} -1 & \phi & \phi \\ \phi & -1 & \phi \\ \phi & \phi & -1 \end{pmatrix} \qquad \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \longrightarrow \{-\hat{e}_1, -\hat{e}_2, -\hat{e}_3\}$$

(we will come back to this)

Displacement - velocity - acceleration vectors

If a vector of components x_k in a certain cs is used to specify the position of a moving point (particle), it is a function of time $x_k = f_k(t)$; in a different cs its components will be called $x_j' = g_j(t)$. Therefore

$$x_j' = l_{kj} x_k = l_{kj} f_k(t) = g_j(t)$$

Velocity is written as

$$v_k = \dot{f}_k(t) = \frac{df_k(t)}{dt} \quad ; \quad v_j' = \dot{g}_j(t) = \frac{dg_j(t)}{dt}$$

but $$v_j' = \frac{d}{dt} x_j' = \frac{d}{dt}(l_{kj} x_k) \underset{\substack{\uparrow \\ l_{kj} \text{ is constant}}}{=} l_{kj} \frac{dx_k}{dt} = l_{kj} v_k$$

So the transformation law for velocities is $\boxed{v_j' = l_{kj} v_k}$

and its inverse transformation is obtained by multiplication times l_{ij} and summation

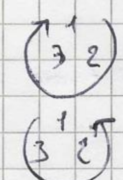
$$l_{ij} v_j' = l_{ij} l_{kj} v_k = \delta_{ik} v_k = v_i \quad \Rightarrow \quad \boxed{v_i = l_{ij} v_j'}$$

We conclude that velocities transform according to the same law as position vectors $\Rightarrow$ velocity is indeed a vector, too. By further time derivation we get to the same result for the acceleration.

Permutation symbol (or Levi - Civita symbol or alternator)

It is defined as follows:

$$\varepsilon_{ijk} \begin{cases} = +1 & \text{for even permutations of } 1\text{-}2\text{-}3 \\ = -1 & \text{for odd permutations of } 1\text{-}2\text{-}3 \\ = \phi & \text{if at least two indices are equal} \end{cases}$$



ε_{ijk} can be very practical at times. We review several theorems in the following (not necessarily proving all of them), following Segel's Chapter 1.

[Definition 3] $\det \underline{\underline{A}} = |A| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$

[Theorem 1] $\det \underline{\underline{A}} = |A| = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$ is also true.

Both expressions are proved by writing explicitly all terms of the sum and then checking the values taken by ε_{ijk} in each of them (only 3 terms remain with +1 and 3 with -1, the others are ϕ).

[Theorem 2] If $\underline{\underline{B}}$ is obtained from $\underline{\underline{A}}$ by interchanging a pair of rows or columns,

$\Rightarrow \det \underline{\underline{B}} = -\det \underline{\underline{A}}$.

(The proof is a matter of permutations.)

[Theorem 3] If $\underline{\underline{A}}$ has two identical rows or colums, $\det \underline{\underline{A}} = \phi$.

Proof: Interchanging the two identical rows/column we get a matrix $\underline{\underline{A}}'$

$\Rightarrow \det \underline{\underline{A}}' = -\det \underline{\underline{A}}$, but $\underline{\underline{A}} = \underline{\underline{A}}'$ so $\det \underline{\underline{A}}' = \det \underline{\underline{A}}$ and $\alpha = -\alpha \Leftrightarrow \alpha = \phi \;\forall \alpha \in \mathbb{R}$.

[Theorem 4] If $\underline{\underline{B}}$ is obtained multiplying a row/column of $\underline{\underline{A}}$ by $c \in \mathbb{R}$, $\det \underline{\underline{B}} = c \det \underline{\underline{A}}$.

[Definition 4] The <u>minor</u> M_{ij} of A_{ij} (element of $\underline{\underline{A}}$) is the determinant of the matrix obtained from $\underline{\underline{A}}$ eliminating the i-th row and j-th column of $\underline{\underline{A}}$. The corresponding <u>cofactor</u> C_{ij} is defined $C_{ij} = (-1)^{i+j} M_{ij}$.

[Theorem 6] $\det \underline{\underline{A}} = \sum_{j=1}^{3} A_{ij} C_{ij} \quad i=1,2,3$; $\det \underline{\underline{A}} = \sum_{i=1}^{3} A_{ij} C_{ij} \quad j=1,2,3$ (not using the summation convention).

[Theorem 7] $A_{pj} C_{ij} = \det \underline{\underline{A}} \cdot \delta_{pi}$; $A_{ip} C_{ij} = \det \underline{\underline{A}} \cdot \delta_{pj}$

[Theorem 8] Cramer's rule: Given the system of linear equations $A_{ij} x_j = b_i$, we have

$\det \underline{\underline{A}} \, x_k = b_i C_{ik}$

Proof: By multiplication of the linear system times the cofactor C_{ik},

$$\underbrace{C_{ik} A_{ij}}_{\det \underline{\underline{A}} \cdot \delta_{kj}} x_j = b_i C_{ik} \quad \Rightarrow \det \underline{\underline{A}}\, \delta_{kj} x_j = \det \underline{\underline{A}}\, x_k = b_i C_{ik}$$

[Theorem 9] $\varepsilon_{rst} \det \underline{\underline{A}} = \varepsilon_{ijk} A_{ir} A_{js} A_{kt} = \varepsilon_{ijk} A_{ri} A_{sj} A_{tk}$

Proof: * If two of the indices r, s, t are equal, the left hand side is zero and the right-hand side is the determinant of a matrix with two identical rows/columns, $\Rightarrow = \phi$.

* For even permutations of (rst), we get a matrix with even swaps of row/column $\Rightarrow$ the determinant (r.h.s.) is the same and $\varepsilon_{rst} = 1$ on the l.h.s..

* For odd permutations, by the same reasoning we conclude that both sides of the relation change their sign.

[Corollary] $\det(\underline{\underline{A}} \cdot \underline{\underline{B}}) = \det \underline{\underline{A}} \cdot \det \underline{\underline{B}}$

Proof: $\det(\underline{\underline{A}} \cdot \underline{\underline{B}}) = \varepsilon_{ijk} (AB)_{i1} (AB)_{j2} (AB)_{k3} =$

by definition of $\det \underline{\underline{B}}$

$$= \varepsilon_{ijk} A_{il} B_{l1} A_{jm} B_{m2} A_{kn} B_{n3} = \det \underline{\underline{A}}\, \varepsilon_{lmn} B_{l1} B_{m2} B_{n3} = \det \underline{\underline{A}} \det \underline{\underline{B}}$$

by Th. 9, $\varepsilon_{ijk} A_{il} A_{jm} A_{kn} = \varepsilon_{lmn} \det \underline{\underline{A}}$

[Theorem 10]
$$\begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix} = \varepsilon_{ijk} \varepsilon_{pqr} \det \underline{\underline{A}}$$

Proof: * If two indices in (ijk) or (pqr) are equal, everything becomes zero.

* If $(ijk) = (123)$ we get

$$\begin{vmatrix} A_{1p} & A_{1q} & A_{1r} \\ A_{2p} & A_{2q} & A_{2r} \\ A_{3p} & A_{3q} & A_{3r} \end{vmatrix} = 1 \cdot \varepsilon_{pqr} \det \underline{\underline{A}}$$

and for $(pqr) = (123)$ or even permutations, we get $\det \underline{\underline{A}} = 1 \cdot \det \underline{\underline{A}}$; while for $(pqr) =$ odd permutations of (123), the l.h.s. differs from $\det \underline{\underline{A}}$ by an odd number of column swaps $\Rightarrow$ changes sign, and the same happens on the r.h.s. since $\varepsilon_{pqr} = -1$.

* The same happens for permutations of (ijk).

[Theorem 11] $\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

Proof: We use Th. 10 with $r=k$ and $A_{ij} = \delta_{ij}$ (identity matrix). That is,

$$\varepsilon_{ijk}\varepsilon_{pqk} \underbrace{\det \mathbb{1}}_{1} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{vmatrix} = \text{expanding the determinant by its third row}$$

$$= \underbrace{\delta_{kp}}_{=1 \Leftrightarrow k=p} \begin{vmatrix} \delta_{iq} & \delta_{ik} \\ \delta_{jq} & \delta_{jk} \end{vmatrix} - \underbrace{\delta_{kq}}_{=1 \Leftrightarrow k=q} \begin{vmatrix} \delta_{ip} & \delta_{ik} \\ \delta_{jp} & \delta_{jk} \end{vmatrix} + \underbrace{\delta_{kk}}_{\text{dummy index: sum} = 3} \begin{vmatrix} \delta_{ip} & \delta_{iq} \\ \delta_{jp} & \delta_{jq} \end{vmatrix} =$$

$$= \underbrace{\begin{vmatrix} \delta_{iq} & \delta_{ip} \\ \delta_{jq} & \delta_{jp} \end{vmatrix}}_{= -\begin{vmatrix} \delta_{ip} & \delta_{iq} \\ \delta_{jp} & \delta_{jq} \end{vmatrix} \text{ by column swap}} - \begin{vmatrix} \delta_{ip} & \delta_{iq} \\ \delta_{jp} & \delta_{jq} \end{vmatrix} + 3 \begin{vmatrix} \delta_{ip} & \delta_{iq} \\ \delta_{jp} & \delta_{jq} \end{vmatrix} = \begin{vmatrix} \delta_{ip} & \delta_{iq} \\ \delta_{jp} & \delta_{jq} \end{vmatrix} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Note: For cyclic (even) permutations one can also write other forms such as

$$\varepsilon_{ijk}\varepsilon_{pqk} = \varepsilon_{ijk}\varepsilon_{kpq} = \varepsilon_{kij}\varepsilon_{kpq} = \varepsilon_{jki}\varepsilon_{qkp} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

ε_{ijk} is very useful when we have to write operations and operators that include vector products. Indeed, if $\bar{a}$, $\bar{b}$ are two vectors, the vector product can be written as

$\bar{a}\times\bar{b} = \varepsilon_{ijk}\,\hat{e}_i\, a_j b_k$ or, in terms of its components, $(\bar{a}\times\bar{b})_i = \varepsilon_{ijk}\, a_j b_k$

It follows that $[\bar{a}\cdot(\bar{b}\times\bar{c})]_i = a_i\cdot(\bar{b}\times\bar{c})_i = \varepsilon_{ijk}\, a_i b_j c_k$

The curl is also conveniently expressed as (using the convention $\partial_j = \partial/\partial x_j$):

$(\text{curl}\,\bar{a})_i = \varepsilon_{ijk}\,\partial_j a_k = \varepsilon_{ijk}\dfrac{\partial a_k}{\partial x_j}$ and we can be convinced of this since, in standard notation,

$$(\text{curl}\,\bar{a})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\,\partial_j a_k = \cancel{\varepsilon_{i11}}\partial_1 a_1 + \varepsilon_{i12}\partial_1 a_2 + \varepsilon_{i13}\partial_1 a_3 +$$
$$+ \varepsilon_{i21}\partial_2 a_1 + \cancel{\varepsilon_{i22}}\partial_2 a_2 + \varepsilon_{i23}\partial_2 a_3 +$$
$$+ \varepsilon_{i31}\partial_3 a_1 + \varepsilon_{i32}\partial_3 a_2 + \cancel{\varepsilon_{i33}}\partial_3 a_3 =$$

$= \text{if } i=1 \rightarrow \varepsilon_{123}\partial_2 a_3 + \varepsilon_{132}\partial_3 a_2 = \partial_2 a_3 - \partial_3 a_2 = \partial a_3/\partial x_2 - \partial a_2/\partial x_3$

$\text{if } i=2 \rightarrow \varepsilon_{213}\partial_1 a_3 + \varepsilon_{231}\partial_3 a_1 = \partial_3 a_1 - \partial_1 a_3$

$\text{if } i=3 \rightarrow \varepsilon_{312}\partial_1 a_3 + \varepsilon_{321}\partial_2 a_1 = \partial_1 a_3 - \partial_2 a_1$

By writing $\text{grad}\,\bar{a} = \partial_j a_i$ and $\text{div}\,\bar{a} = \partial_i a_i$ we can also verify that $\text{curl}(\text{curl}\,\bar{a}) = \text{grad}(\text{div}\,\bar{a}) - \nabla^2 \bar{a}$; indeed for the i-th component

$$[\text{curl}(\text{curl}\,\bar{a})]_i = \varepsilon_{ijl}\partial_j(\text{curl}\,\bar{a})_l = \varepsilon_{ijl}\partial_j(\varepsilon_{lhk}\partial_h a_k) =$$

$$= \varepsilon_{ijl}\varepsilon_{lhk}\partial_j\partial_h a_k \underset{\substack{\uparrow \\ \text{with cyclic permutation of } \varepsilon_{lhk}}}{=} \varepsilon_{ijl}\varepsilon_{hkl}\partial_j\partial_h a_k = (\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh})\partial_j\partial_h a_k =$$

$$= \delta_{ih}\delta_{jk}\partial_j\partial_h a_k - \delta_{ik}\delta_{jh}\partial_j\partial_h a_k =$$

$$= \partial_j\partial_i a_j - \partial_j\partial_j a_i = \partial_i\underbrace{\partial_j a_j}_{\text{div}} - \underbrace{\partial_j\partial_j}_{\nabla^2} a_i = [\text{grad}(\text{div}\,\bar{a})]_i - (\nabla^2\bar{a})_i$$

Another useful property descends from Theorem 9 ($\rightarrow \varepsilon_{rst}\det \underline{\underline{A}} = \varepsilon_{ijk}A_{ir}A_{js}A_{kt}$):

$$\boxed{\varepsilon_{pjk}A_{js}A_{kt} = (\det\underline{\underline{A}})A^{-1}_{rp}\varepsilon_{rst}}$$

Proof: $(\det\underline{\underline{A}})A^{-1}_{rp}\varepsilon_{rst} = \varepsilon_{ijk}\underbrace{A_{ir}A^{-1}_{rp}}_{\delta_{ip}\text{ (an element of the identity matrix: } \underline{\underline{A}}\cdot\underline{\underline{A}}^{-1} = \mathbb{1})}A_{js}A_{kt} = \varepsilon_{ijk}\delta_{ip}A_{js}A_{kt} = \varepsilon_{pjk}A_{js}A_{kt}$

Notice that if $\underline{\underline{A}} = \underline{\underline{l}}$ orthogonal matrix (i.e. a CS transformation), by Theorem 9

$$\varepsilon_{rst}\det\underline{\underline{l}} = \varepsilon_{ijk}l_{ir}l_{js}l_{kt}$$

$$\Rightarrow \varepsilon_{rst} = (\det\underline{\underline{l}})^{-1}\varepsilon_{ijk}l_{ir}l_{js}l_{kt} \quad \text{but} \quad (\det\underline{\underline{l}})^{-1} = \det\underline{\underline{l}} \quad (\pm 1)$$

$$\Rightarrow \boxed{\varepsilon_{rst} = (\det\underline{\underline{l}})\varepsilon_{ijk}l_{ir}l_{js}l_{kt}} \quad \underline{\text{transformation law for } \varepsilon_{ijk}!}$$

This law looks a bit like the one for a (polar) vector, differing by the presence of $\det\underline{\underline{l}} = \pm 1$; we will soon see this is the transformation law for a <u>pseudotensor</u>, much like there is a transformation law for <u>pseudovectors</u> (or <u>axial vectors</u>) differing from the law for polar vectors by the presence of a factor $\det\underline{\underline{l}}$ (there is an actual difference only when $\det\underline{\underline{l}} = -1$, i.e. upon reflection).

The property above is also used to obtain the transformation law of an important vector quantity, i.e. the vector product between two polar vectors $\bar{a}, \bar{b}$. Its i-th component is

$(\bar{a}\times\bar{b})_i = \varepsilon_{irs} a_r b_s$ in a certain cs; introducing another (primed) cs

$$(\bar{a}\times\bar{b})_i = \varepsilon_{irs} a_r b_s = \boxed{\varepsilon_{irs}\, l_{rp}\, a'_p\, l_{sq}}\, b'_q = (\det \underline{\underline{l}})\, l^{-1}_{ki}\, \underbrace{\varepsilon_{kpq} a'_p b'_q}_{(\bar{a}\times\bar{b})'_k} =$$

$$\varepsilon_{irs}\, l_{rp}\, l_{sq} = (\det \underline{\underline{l}})\, l^{-1}_{ki}\, \varepsilon_{kpq} \quad = \text{ since } l^{-1}_{ki} = l_{ik}$$

$$= (\det \underline{\underline{l}})\, l_{ik}\, (\bar{a}\times\bar{b})'_k$$

$$\Rightarrow \boxed{(\bar{a}\times\bar{b})_i = (\det \underline{\underline{l}})\, l_{ik}\, (\bar{a}\times\bar{b})'_k}$$ that is the inverse transformation

of $$\boxed{(\bar{a}\times\bar{b})'_j = (\det \underline{\underline{l}})\, l_{ij}\, (\bar{a}\times\bar{b})_i}$$

Notice that this is not exactly the law for a polar vector by the presence of $(\det \underline{\underline{l}}) \Rightarrow$

$\underline{v'_j = (\det \underline{\underline{l}})\, l_{ij}\, v_i}$ is the transformation law for a PSEUDOVECTOR or AXIAL VECTOR, by which we prove that the vector product of two polar vectors is an axial vector.

[One can prove that polar vector $\times$ axial vector $\rightarrow$ polar vector;
axial vector $\times$ axial vector $\rightarrow$ axial vector.]

Let us review the concepts of polar and axial vector.

POLAR VECTOR: Its components transform in their opposite by inversion of coordinate axes (reflection) $\Rightarrow$ it <u>stays the same</u>.

AXIAL VECTOR: Its components are invariant under inversion of the coordinate axes $\Rightarrow$ it <u>is reversed</u>.

Example: Assume a certain coordinate system $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and reflect $\hat{e}_1 \rightarrow \hat{e}'_1 = -\hat{e}_1$; the orthogonal matrix of this transformation is

$$\underline{\underline{l}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{indeed } \hat{e}'_1 = \underline{\underline{l}}\cdot\hat{e}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\hat{e}_1$$

$$\hat{e}'_2 = \underline{\underline{l}}\cdot\hat{e}_2 = \hat{e}_2$$

$$\hat{e}'_3 = \underline{\underline{l}}\cdot\hat{e}_3 = \hat{e}_3$$

Let us say what happens to a polar vector and to an axial one.

⊛ Have $\bar{a}$ polar vector; $\Rightarrow a_j' = l_{ij} a_i = l_{1j} a_1 + l_{2j} a_2 + l_{3j} a_3$ in CS' primed system

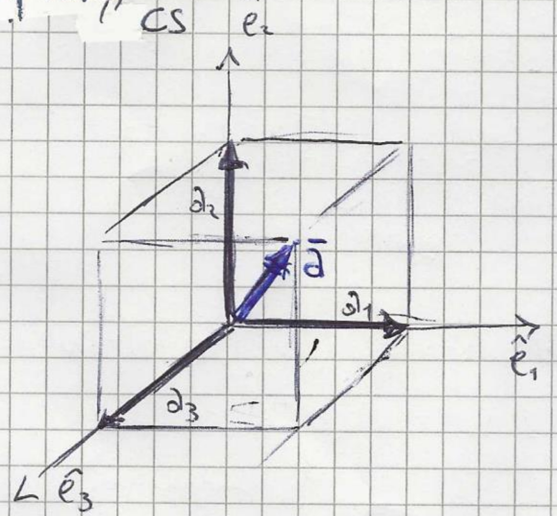
when $j = 1$ $\quad a_1' = l_{11} a_1 + \cancel{l_{21} a_2} + \cancel{l_{31} a_3} = l_{11} a_1 = -a_1$

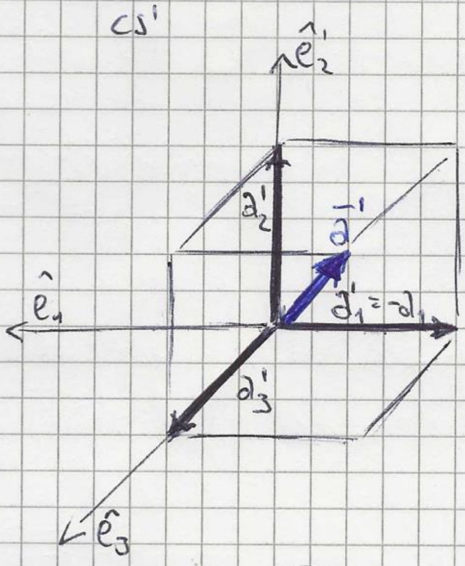
$j = 2$ $\quad a_2' = \cancel{l_{12} a_1} + l_{22} a_2 + \cancel{l_{32} a_3} = l_{22} a_2 = a_2$

$j = 3$ $\quad a_3' = \cancel{l_{13} a_1} + \cancel{l_{23} a_2} + l_{33} a_3 = l_{33} a_3 = a_3$

(only the component along the reversed axis is reversed)

Graphically,





$\bar{a}'$ is still the same as $\bar{a}$

⊛ Now let $\bar{a}$ be an axial vector; $\Rightarrow a_j' = \det \underline{\underline{l}}\, l_{ij} a_i = (-1) \cdot (l_{1j} a_1 + l_{2j} a_2 + l_{3j} a_3)$

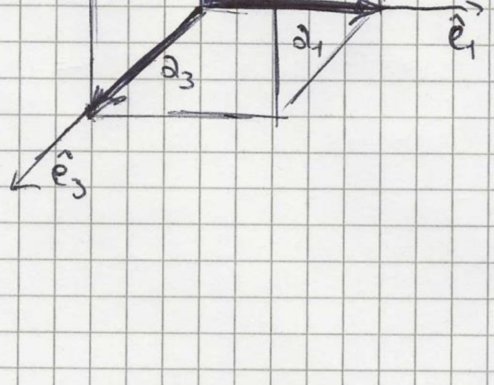
$\Rightarrow a_1' = -1 \cdot l_{11} a_1 = a_1$

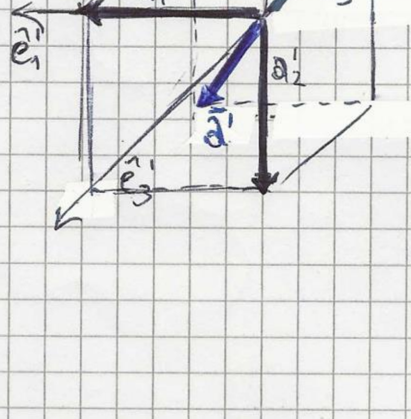
$a_2' = -1 \cdot l_{22} a_2 = -a_2$

$a_3' = -1 \cdot l_{33} a_3 = -a_3$

(the two components along the non-reversed axes are reversed)

Graphically,





$\bar{a}$ is reflected into $\bar{a}'$