Equazione di Navier-Stokes in coordinate non cartesiane

L'eq. di N.-S. che descrive la dinamica di un fluido viscoso incomprimibile è scritta nella forma generale

$$\frac{D\bar{v}}{Dt} = \frac{1}{\rho}\,\mathrm{div}(\underline{\underline{\sigma}}) \left(-\mathrm{grad}\, u\right)$$

In coordinate cartesiane, conoscendo il tensore $\underline{\underline{\sigma}}$ cartesiano l'abbiamo potuta specificare come

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\mathrm{grad})\,\bar{v} = \nu \nabla^2 \bar{v} - \frac{1}{\rho}\,\mathrm{grad}\, p \quad \left(-\mathrm{grad}\, u\right)$$

con gli operatori differenziali nella facile forma cartesiana:

$$\mathrm{grad} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \quad ; \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Accenniamo a come si svolgono alcuni dei calcoli per ottenere le forme dell'eq. di N.-S. in altri sistemi di coordinate ortogonali di ampio uso, riportando in particolare i risultati per le coordinate cilindriche e sferiche.

Coordinate cilindriche

① Dobbiamo innanzitutto calcolare il termine $(\bar{v}\cdot\mathrm{grad})\,\bar{v}$ in coord. cilindriche. L'operatore $(\bar{v}\cdot\mathrm{grad})$ è semplicemente espresso come

$$\bar{v}\cdot(\mathrm{grad})_{cyl} = v_r \frac{\partial}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial}{\partial \vartheta} + v_z \frac{\partial}{\partial z}$$

Applicandolo a $\bar{v}$, la differenziazione del vettore $\bar{v}$ richiede che si differenzino anche i versori, che non sono fissi come quelli cartesiani. Infatti

$$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\vartheta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \phi \; ; \quad \frac{\partial \hat{e}_z}{\partial \vartheta} = \phi \; ; \quad \frac{\partial \hat{e}_r}{\partial z} = \frac{\partial \hat{e}_\vartheta}{\partial z} = \frac{\partial \hat{e}_z}{\partial z} = \phi$$

ma $\underline{\dfrac{\partial \hat{e}_r}{\partial \vartheta} = \hat{e}_\vartheta}$; $\underline{\dfrac{\partial \hat{e}_\vartheta}{\partial \vartheta} = -\hat{e}_r}$

Perciò

$$v_r \frac{\partial \bar{v}}{\partial r} = v_r \left[\frac{\partial v_r}{\partial r}\hat{e}_r + v_r \cancel{\frac{\partial \hat{e}_r}{\partial r}} + \frac{\partial v_\vartheta}{\partial r}\hat{e}_\vartheta + v_\vartheta \cancel{\frac{\partial \hat{e}_\vartheta}{\partial r}} + \frac{\partial v_z}{\partial r}\hat{e}_z + v_z \cancel{\frac{\partial \hat{e}_z}{\partial r}} \right] =$$

$$= v_r \left[\frac{\partial v_r}{\partial r}\hat{e}_r + \frac{\partial v_\vartheta}{\partial r}\hat{e}_\vartheta + \frac{\partial v_z}{\partial r}\hat{e}_z \right]$$

$$\frac{v_\vartheta}{r}\frac{\partial \bar{v}}{\partial \vartheta} = \frac{v_\vartheta}{r}\left[\frac{\partial v_r}{\partial \vartheta}\hat{e}_r + v_r\frac{\partial \hat{e}_r}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial \vartheta}\hat{e}_\vartheta + v_\vartheta\frac{\partial \hat{e}_\vartheta}{\partial \vartheta} + \frac{\partial v_z}{\partial \vartheta}\hat{e}_z + \cancel{v_z\frac{\partial \hat{e}_z}{\partial \vartheta}}\right] =$$

$$= \frac{v_\vartheta}{r}\left[\frac{\partial v_r}{\partial \vartheta}\hat{e}_r + v_r\hat{e}_\vartheta + \frac{\partial v_\vartheta}{\partial \vartheta}\hat{e}_\vartheta - v_\vartheta\hat{e}_r + \frac{\partial v_z}{\partial \vartheta}\hat{e}_z\right]$$

$$v_z\frac{\partial \bar{v}}{\partial z} = v_z\left[\frac{\partial v_r}{\partial z}\hat{e}_r + \cancel{v_r\frac{\partial \hat{e}_r}{\partial z}} + \frac{\partial v_\vartheta}{\partial z}\hat{e}_\vartheta + \cancel{v_\vartheta\frac{\partial \hat{e}_\vartheta}{\partial z}} + \frac{\partial v_z}{\partial z}\hat{e}_z + \cancel{v_z\frac{\partial \hat{e}_z}{\partial z}}\right] =$$

$$= v_z\left[\frac{\partial v_r}{\partial z}\hat{e}_r + \frac{\partial v_\vartheta}{\partial z}\hat{e}_\vartheta + \frac{\partial v_z}{\partial z}\hat{e}_z\right]$$

Dunque risultano due "pezzi in più", $-\frac{v_\vartheta^2}{r}\hat{e}_r$ e $\frac{v_r v_\vartheta}{r}\hat{e}_\vartheta$.

② Per esprimere il termine $\operatorname{div}\underline{\underline{\sigma}}$ al secondo membro è intanto necessario scrivere il tensore degli sforzi in coordinate cilindriche. Esso è, in generale,

$\sigma_{ij} = \sigma'_{ij} - p\delta_{ij}$ o in forma equivalente $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}' - p\underline{\underline{1}}$

con la parte deviatorica che, per flusso incomprimibile, si riduce a

$\sigma'_{ij} = \eta(\partial_j v_i + \partial_i v_j)$ o equivalentemente $\underline{\underline{\sigma}}' = \eta(\operatorname{grad}\bar{v} + (\operatorname{grad}\bar{v})^T)$

Il gradiente di $\bar{v}$ si può esprimere come $\bar{\nabla}\bar{v} = (\bar{\nabla}\otimes\bar{v})^T$, dove, come sopra, si deve considerare anche di derivare i versori: $\left[\Rightarrow (\bar{\nabla}\bar{v})^T = \bar{\nabla}\otimes\bar{v}\right]$

$$(\bar{\nabla}\bar{v})^T = \bar{\nabla}\otimes\bar{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\vartheta\frac{1}{r}\frac{\partial}{\partial \vartheta} + \hat{e}_z\frac{\partial}{\partial z}\right)\otimes(v_r\hat{e}_r + v_\vartheta\hat{e}_\vartheta + v_z\hat{e}_z) =$$

$$= \hat{e}_r\otimes\frac{\partial}{\partial r}(v_r\hat{e}_r) + \hat{e}_r\otimes\frac{\partial}{\partial r}(v_\vartheta\hat{e}_\vartheta) + \hat{e}_r\otimes\frac{\partial}{\partial r}(v_z\hat{e}_z) +$$

$$+ \hat{e}_\vartheta\otimes\frac{1}{r}\frac{\partial}{\partial \vartheta}(v_r\hat{e}_r) + \hat{e}_\vartheta\otimes\frac{1}{r}\frac{\partial}{\partial \vartheta}(v_\vartheta\hat{e}_\vartheta) + \hat{e}_\vartheta\otimes\frac{1}{r}\frac{\partial}{\partial \vartheta}(v_z\hat{e}_z) +$$

$$+ \hat{e}_z\otimes\frac{\partial}{\partial z}(v_r\hat{e}_r) + \hat{e}_z\otimes\frac{\partial}{\partial z}(v_\vartheta\hat{e}_\vartheta) + \hat{e}_z\otimes\frac{\partial}{\partial z}(v_z\hat{e}_z) =$$

$$= \frac{\partial v_r}{\partial r}\hat{e}_r\otimes\hat{e}_r + \frac{\partial v_\vartheta}{\partial r}\hat{e}_r\otimes\hat{e}_\vartheta + \frac{\partial v_z}{\partial r}\hat{e}_r\otimes\hat{e}_z +$$

$$+ \frac{1}{r}\frac{\partial v_r}{\partial \vartheta}\hat{e}_\vartheta\otimes\hat{e}_r + \frac{v_r}{r}\hat{e}_\vartheta\otimes\hat{e}_\vartheta + \frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta}\hat{e}_\vartheta\otimes\hat{e}_\vartheta - \frac{v_\vartheta}{r}\hat{e}_\vartheta\otimes\hat{e}_r + \frac{1}{r}\frac{\partial v_z}{\partial \vartheta}\hat{e}_\vartheta\otimes\hat{e}_z +$$

$$+ \frac{\partial v_r}{\partial z}\hat{e}_z\otimes\hat{e}_r + \frac{\partial v_\vartheta}{\partial z}\hat{e}_z\otimes\hat{e}_\vartheta + \frac{\partial v_z}{\partial z}\hat{e}_z\otimes\hat{e}_z$$

e in forma matriciale

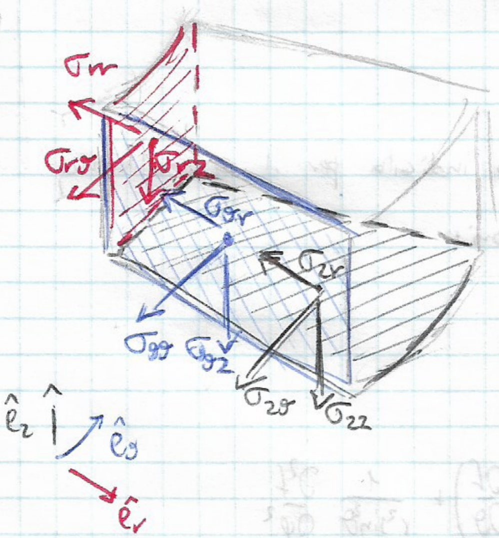
$$(\bar\nabla \bar v)^T = \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\vartheta}{\partial r} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r} & \frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_r}{r} & \frac{1}{r}\frac{\partial v_z}{\partial \vartheta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\vartheta}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

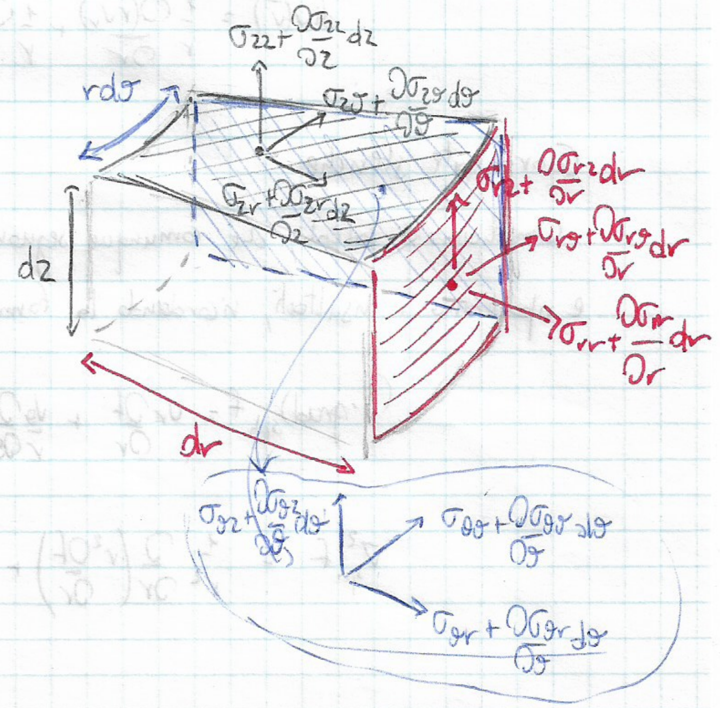
Dunque infine, poiché $\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{1}}} + \underline{\underline{\sigma}}' = -p\underline{\underline{\mathbb{1}}} + \eta\left(\bar\nabla\bar v + (\bar\nabla \bar v)^T\right)$,

$$\boxed{\begin{aligned} &\sigma_{rr} = -p + 2\eta\frac{\partial v_r}{\partial r} && \sigma_{r\vartheta} = \sigma_{\vartheta r} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial r} - \frac{v_\vartheta}{r}\right) \\ &\sigma_{\vartheta\vartheta} = -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_r}{r}\right) && \sigma_{\vartheta z} = \sigma_{z\vartheta} = \eta\left(\frac{1}{r}\frac{\partial v_z}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial z}\right) \\ &\sigma_{zz} = -p + 2\eta\frac{\partial v_z}{\partial z} && \sigma_{rz} = \sigma_{zr} = \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \end{aligned}}$$

③ Resta da esprimere la divergenza di $\underline{\underline{\sigma}}$ in coordinate cilindriche. L'espressione della divergenza, così come si fa anche per la divergenza di un campo vettoriale, si ottiene calcolando il flusso attraverso un volumetto infinitesimo con le facce normali ai versori della base $(\hat e_r, \hat e_\vartheta, \hat e_z)$. Si sommano dunque per ogni direzione tutti gli sforzi lungo tale direzione che si trovano sulle 3 coppie di facce. Per volumetto di spigoli infinitesimi, gli sforzi per uno spostamento infinitesimo lungo una direzione si esprimono come sviluppo al primo ordine.

Graficamente





Omettiamo il calcolo completo, il risultato è

$$\operatorname{div}\underline{\underline{\sigma}} = \left(\frac{\sigma_{rr}}{r} + \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{rz}}{\partial z} - \frac{\sigma_{\vartheta\vartheta}}{r}\right)\hat{e}_r +$$

$$+\left(\frac{2\sigma_{r\vartheta}}{r} + \frac{\partial\sigma_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{\vartheta z}}{\partial z}\right)\hat{e}_\vartheta +$$

$$+\left(\frac{\sigma_{rz}}{r} + \frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\vartheta z}}{\partial\vartheta} + \frac{\partial\sigma_{zz}}{\partial z}\right)\hat{e}_z$$

In conclusione si può scrivere l'eq. di Navier-Stokes in componenti cilindriche:

$$\boxed{\begin{array}{ll}
r & \dfrac{\partial v_r}{\partial t} + (\bar{v}\cdot\operatorname{grad})v_r - \dfrac{v_\vartheta^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \dfrac{2}{r^2}\dfrac{\partial v_\vartheta}{\partial\vartheta} - \dfrac{v_r}{r^2}\right) \\[2ex]
\vartheta & \dfrac{\partial v_\vartheta}{\partial t} + (\bar{v}\cdot\operatorname{grad})v_\vartheta + \dfrac{v_r v_\vartheta}{r} = -\dfrac{1}{\rho r}\dfrac{\partial p}{\partial\vartheta} + \nu\left(\nabla^2 v_\vartheta + \dfrac{2}{r^2}\dfrac{\partial v_r}{\partial\vartheta} - \dfrac{v_\vartheta}{r^2}\right) \\[2ex]
z & \dfrac{\partial v_z}{\partial t} + (\bar{v}\cdot\operatorname{grad})v_z = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \nu\nabla^2 v_z
\end{array}}$$

(avendo usato le espressioni esplicite delle componenti di $\underline{\underline{\sigma}}$ in $\operatorname{div}\underline{\underline{\sigma}}$)

dove il laplaciano in coordinate cilindriche è

$$\nabla^2_{cyl} f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\vartheta^2} + \frac{\partial^2 f}{\partial z^2}$$

Si ha poi in aggiunta la condizione di incomprimibilità

$$\operatorname{div}(\bar{v}) = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta} + \frac{\partial v_z}{\partial z} = \emptyset$$

Coordinate sferiche

Omettiamo i calcoli, che comunque seguono la traccia indicata per il caso cilindrico, e riportiamo i risultati, ricordando la forma degli operatori:

$$(\bar{v}\cdot\operatorname{grad})_{sph} f = v_r\frac{\partial f}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial f}{\partial\vartheta} + \frac{v_\varphi}{r\sin\vartheta}\frac{\partial f}{\partial\varphi}$$

$$\nabla^2_{sph} f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}$$

Il tensore degli sforzi ha componenti

$$\sigma_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r} \qquad \sigma_{r\theta} = \sigma_{\theta r} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\sigma_{\theta\theta} = -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \qquad \sigma_{\theta\varphi} = \sigma_{\varphi\theta} = \eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \theta} - \frac{v_\varphi \cot g\,\theta}{r}\right)$$

$$\sigma_{\varphi\varphi} = -p + 2\eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\theta \cot g\,\theta}{r}\right) \qquad \sigma_{r\varphi} = \sigma_{\varphi r} = \eta\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)$$

L'eq. di Navier-Stokes ha componenti:

$$(r) \quad \frac{\partial v_r}{\partial t} + (\bar{v}\cdot \mathrm{grad})v_r - \frac{v_\theta^2 + v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 v_r - \frac{2}{r^2\sin^2\theta}\frac{\partial(v_\theta \sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{2v_r}{r^2}\right]$$

$$(\theta) \quad \frac{\partial v_\theta}{\partial t} + (\bar{v}\cdot \mathrm{grad})v_\theta + \frac{v_r v_\theta}{r} - \frac{v_\varphi^2 \cot g\,\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\nabla^2 v_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\right]$$

$$(\varphi) \quad \frac{\partial v_\varphi}{\partial t} + (\bar{v}\cdot \mathrm{grad})v_\varphi + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cot g\,\theta}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \varphi} + \nu\left[\nabla^2 v_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{r^2\sin^2\theta}\right]$$

a cui si aggiunge la condizione di incomprimibilità

$$\mathrm{div}(\bar{v}) = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} = 0$$