Analisi vettoriale

1 Operatori differenziali

1.1 Coordinate cartesiane - (x, y, z)

$$\mathbf{r} = (x, y, z)$$

$$d\mathbf{l} = dx\,\hat{\mathbf{e}}_x + dy\,\hat{\mathbf{e}}_y + dz\,\hat{\mathbf{e}}_z$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial f}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z$$

$$\vec{\nabla} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\vec{\nabla} \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{e}}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{e}}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{e}}_z$$

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

1.2 Coordinate cilindriche - (r, θ, z)

$$\mathbf{r} = (r\cos\theta, r\sin\theta, z)$$

$$d\mathbf{l} = dr\,\hat{\mathbf{e}}_r + r d\theta\,\hat{\mathbf{e}}_\theta + dz\,\hat{\mathbf{e}}_z$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z$$

$$\vec{\nabla} \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\vec{\nabla} \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{\mathbf{e}}_z$$

$$\vec{\nabla}^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

1.3 Coordinate sferiche - (r, θ, ϕ)

$$\mathbf{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

$$d\mathbf{l} = dr\,\hat{\mathbf{e}}_r + r d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta d\phi\,\hat{\mathbf{e}}_\phi$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi$$

$$\vec{\nabla} \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (F_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\vec{\nabla} \times \mathbf{F} = \frac{1}{r\sin\theta}\left(\frac{\partial (F_\phi \sin\theta)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right)\hat{\mathbf{e}}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial (rF_\phi)}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{\mathbf{e}}_\phi$$

$$\vec{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

2 Identità differenziali

$$\vec{\nabla}(\phi+\psi)=\vec{\nabla}\phi+\vec{\nabla}\psi$$

$$\vec{\nabla}(\phi\psi)=\psi\vec{\nabla}\phi+\phi\vec{\nabla}\psi$$

$$\vec{\nabla}\cdot(\mathbf{F}+\mathbf{G})=\vec{\nabla}\cdot\mathbf{F}+\vec{\nabla}\cdot\mathbf{G}$$

$$\vec{\nabla}\times(\mathbf{F}+\mathbf{G})=\vec{\nabla}\times\mathbf{F}+\vec{\nabla}\times\mathbf{G}$$

$$\vec{\nabla}(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\vec{\nabla})\mathbf{G}+(\mathbf{G}\cdot\vec{\nabla})\mathbf{F}+\mathbf{F}\times(\vec{\nabla}\times\mathbf{G})+\mathbf{G}\times(\vec{\nabla}\times\mathbf{F})$$

$$\vec{\nabla}\cdot(\phi\mathbf{F})=\phi\vec{\nabla}\cdot\mathbf{F}+\mathbf{F}\cdot\vec{\nabla}\phi$$

$$\vec{\nabla}\cdot(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot(\vec{\nabla}\times\mathbf{F})-\mathbf{F}\cdot(\vec{\nabla}\times\mathbf{G})$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\mathbf{F})=0$$

$$\vec{\nabla}\times(\phi\mathbf{F})=\phi\vec{\nabla}\times\mathbf{F}+\vec{\nabla}\phi\times\mathbf{F}$$

$$\vec{\nabla}\times(\mathbf{F}\times\mathbf{G})=\mathbf{F}(\vec{\nabla}\cdot\mathbf{G})-\mathbf{G}(\vec{\nabla}\cdot\mathbf{F})+(\mathbf{G}\cdot\vec{\nabla})\mathbf{F}-(\mathbf{F}\cdot\vec{\nabla})\mathbf{G}$$

$$\vec{\nabla}\times\vec{\nabla}\times\mathbf{F}=\vec{\nabla}(\vec{\nabla}\cdot\mathbf{F})-\vec{\nabla}^2\mathbf{F}$$

$$\vec{\nabla}\times\vec{\nabla}\phi=0$$

$$\oint_S \mathbf{F}\cdot d\mathbf{a}=\int_V \vec{\nabla}\cdot\mathbf{F}\,dV$$

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{l}=\int_S(\vec{\nabla}\times\mathbf{F})\cdot d\mathbf{a}$$

$$\oint_S \phi\, d\mathbf{a}=\int_V \vec{\nabla}\phi\, dV$$

$$\oint_S \mathbf{F}(\mathbf{G}\cdot\hat{\mathbf{n}})\,da=\int_V \mathbf{F}(\vec{\nabla}\cdot\mathbf{G})\,dV+\int_V(\mathbf{G}\cdot\vec{\nabla})\mathbf{F}\,dV$$

$$\oint_S(\hat{\mathbf{n}}\times\mathbf{F})\,da=\int_V(\vec{\nabla}\times\mathbf{F})\,dV$$

$$\oint_\Gamma \phi\, d\mathbf{l}=\int_S(\hat{\mathbf{n}}\times\vec{\nabla}\phi)\,da$$