APPENDIX IV

VECTOR DIFFERENTIAL OPERATORS

1. *Rectangular Coordinates*

$$\text{grad } \varphi = \mathbf{i}\frac{\partial \varphi}{\partial x} + \mathbf{j}\frac{\partial \varphi}{\partial y} + \mathbf{k}\frac{\partial \varphi}{\partial z},$$

$$\text{div } \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

$$\text{curl } \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

2. *Cylindrical Coordinates*

$$\text{grad } \varphi = \mathbf{a}_r\frac{\partial \varphi}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \mathbf{k}\frac{\partial \varphi}{\partial z},$$

$$\text{div } \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z},$$

$$\text{curl } \mathbf{F} = \mathbf{a}_r\left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right) + \mathbf{a}_\theta\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) + \mathbf{k}\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right).$$

3. *Spherical Coordinates*

$$\text{grad } \varphi = \mathbf{a}_r\frac{\partial \varphi}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \mathbf{a}_\phi\frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi},$$

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi},$$

$$\text{curl } \mathbf{F} = \mathbf{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi\sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right] + \mathbf{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial(rF_\phi)}{\partial r}\right] + \mathbf{a}_\phi\frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right].$$

Rectangular coordinates:

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}. \qquad (3\text{–}6)$$

Spherical coordinates:

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}. \qquad (3\text{–}7)$$

Cylindrical coordinates:

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}. \qquad (3\text{–}8)$$

Another way in which the application of the vector differential operators may be extended is to apply them to various products of vectors and scalars. There are many possible combinations of differential operators and products; those of most interest are tabulated in Table 1–1. These identities may be readily verified in rectangular coordinates, which is sufficient to assure their validity in any coordinate system.

TABLE 1–1

FORMULAS FROM VECTOR ANALYSIS INVOLVING DIFFERENTIAL OPERATORS

(I–1)	$\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$
(I–2)	$\nabla\varphi\psi = \varphi\nabla\psi + \psi\nabla\varphi$
(I–3)	$\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$
(I–4)	$\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$
(I–5)	$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times\operatorname{curl}\mathbf{G} + \mathbf{G}\times\operatorname{curl}\mathbf{F}$
(I–6)	$\operatorname{div}\varphi\mathbf{F} = \varphi\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla\varphi$
(I–7)	$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$
(I–8)	$\operatorname{div}\operatorname{curl}\mathbf{F} = 0$
(I–9)	$\operatorname{curl}\varphi\mathbf{F} = \varphi\operatorname{curl}\mathbf{F} + \nabla\varphi\times\mathbf{F}$
(I–10)	$\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$
(I–11)	$\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$
(I–12)	$\operatorname{curl}\nabla\varphi = 0$
(I–13)	$\oint_S \mathbf{F}\cdot\mathbf{n}\,da = \int_V \operatorname{div}\mathbf{F}\,dv$
(I–14)	$\oint_C \mathbf{F}\cdot d\mathbf{l} = \int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,da$
(I–15)	$\oint_S \varphi\mathbf{n}\,da = \int_V \nabla\varphi\,dv$
(I–16)	$\oint_S \mathbf{F}(\mathbf{G}\cdot\mathbf{n})\,da = \int_V \mathbf{F}\operatorname{div}\mathbf{G}\,dv + \int_V (\mathbf{G}\cdot\nabla)\mathbf{F}\,dv$
(I–17)	$\oint_S \mathbf{n}\times\mathbf{F}\,da = \int_V \operatorname{curl}\mathbf{F}\,dv$
(I–18)	$\oint_C \varphi\,d\mathbf{l} = \int_S \mathbf{n}\times\nabla\varphi\,da$

There are several possibilities for the extension of the divergence theorem and of Stokes' theorem. The most interesting of these is Green's theorem, which is

$$\int_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\,dv = \oint_S (\psi\operatorname{grad}\varphi - \varphi\operatorname{grad}\psi)\cdot\mathbf{n}\,da. \tag{1–52}$$

This theorem follows from the application of the divergence theorem to the vector

$$\mathbf{F} = \psi\operatorname{grad}\varphi - \varphi\operatorname{grad}\psi. \tag{1–53}$$

Using this F in the divergence theorem, we obtain