

Energy transport in gravity waves

let us consider a gravity wave propagating along the x direction in an ideal fluid, within an incompressible potential flow approximation. The system is y -invariant, z vertical axis, and we know

$$\left. \begin{aligned} \varphi(x, z, t) &= A \cosh[k(z+h)] \cos(kx - \omega t) \\ \zeta(x, t) &= -\frac{1}{g} \omega A \cosh(kh) \sin(kx - \omega t) \end{aligned} \right\} \text{for a wave in a basin with bottom } z = -h \text{ and vacuum (negligible fluid) on top}$$

Let us consider a surface through which we want to calculate the energy flux. This is a plane with normal unit vector \hat{e}_x , width L in the y -direction, and height measured between the bottom $z = -h$ to the top of the fluid layer $z = \zeta$; let us call this surface $S(\zeta)$. When we studied the energy flux we found out the flux density is $\rho(E_m + \omega)\bar{v}$, that is here $\rho\left(\frac{1}{2}v^2 + gz + p/\rho + E\right)\bar{v}$ and by ignoring E (conserved in an incompressible flow) the flux through $S(\zeta)$ results

$$Q^E(S) = \int_S \rho\left(\frac{1}{2}v^2 + gz + p/\rho\right)\bar{v} \cdot d\bar{S} = \rho \int_S \left(\frac{1}{2}v^2 + gz + p/\rho\right) v_x dy dz$$

Making use of the generalized Bernoulli's equation applied between a generic z and ζ ,

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz + \frac{\partial\varphi}{\partial t} = \frac{1}{2}v^2(\zeta) + \frac{p_0}{\rho} + g\zeta + \frac{\partial\varphi(\zeta)}{\partial t};$$

pressure at the free surface is a constant which we can set as $p_0 = \phi$ (negligible for very light upper fluid); the kinetic terms are quadratic, hence negligible in a linear approximation; finally, the interface boundary condition yields

$$g\zeta(x, t) + \frac{\partial}{\partial t} \varphi(x, \zeta, t) = \phi \quad \text{and therefore we simplify the equation above into}$$

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = -\frac{\partial\varphi}{\partial t}$$

$$\Rightarrow Q^E(S) = -\int_y^{y+L} \int_{z=-h}^{z=\zeta} \rho \frac{\partial\varphi}{\partial t} v_x dy dz = -\rho L \int_{z=-h}^{z=\zeta} \frac{\partial\varphi}{\partial t} v_x dz = -\rho L \int_{-h}^{\zeta} \frac{\partial\varphi}{\partial t} \frac{\partial\varphi}{\partial x} dz$$

Now let us split the integral over two subintervals, $[-h, \phi]$ and $[\phi, \zeta]$; the second term, discarding higher-order infinitesimal contribution, can be approximated using the value in $z = \phi$ of the integrand, so that

$$Q^E(S) = -\rho L \int_{-h}^{\phi} \frac{\partial\varphi}{\partial t} \frac{\partial\varphi}{\partial x} dz - \rho L \frac{\partial\varphi}{\partial t}(x, \phi, t) \frac{\partial\varphi}{\partial x}(x, \phi, t) \zeta(x, t)$$

The first term is second order in magnitude, while the second one is a third-order contribution. Furthermore, we are interested in calculating a period-average quantity, and one could see that the second term, as a product of three sinusoidal function, has a zero average. So let us make a time average over a period T :

$$\langle Q^E(S) \rangle = -PL \frac{1}{T} \int_{t_0}^{t_0+T} \int_{z=-h}^{z=\phi} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x} dz dt \quad \text{and plugging in the explicit form of } \varphi$$

$$\frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x} = \omega A \cosh[k(z+h)] \sin(kx - \omega t) \cdot (-k) A \cosh[k(z+h)] \sin(kx - \omega t) =$$

$$= -k\omega A^2 \cosh^2[k(z+h)] \sin^2(kx - \omega t)$$

and since $\langle \sin(kx - \omega t) \rangle = \frac{1}{2}$, $\int \cosh^2(x) dx = \frac{1}{4a} \sinh(2ax) + \frac{x}{2} + C$

$$\langle Q^E(S) \rangle = \frac{1}{2} PL k \omega A^2 \int_{-h}^{\phi} \cosh^2[k(z+h)] dz = \frac{1}{2} PL k \omega A^2 \left[\frac{1}{4k} \sinh(2k(z+h)) + \frac{z+h}{2} \right]_{-h}^{\phi} =$$

$$= \frac{1}{8} PL \omega A^2 [2kh + \sinh(2kh)]$$

Now let us calculate the time-averaged amount of mechanical energy stored in a region $R(\zeta)$ that is just upstream of $S(\zeta)$ and extends in the x direction of a quantity $\Delta x = \lambda$ (one wavelength). Any point within $R(\zeta)$ is a point (x, y, z) such that

$$\begin{cases} x_0 < x < x_0 + \lambda \\ y_0 < y < y_0 + L \\ -h < z < \zeta(x, t) \end{cases}$$

We also call $R(\phi)$ a subregion of $R(\zeta)$ of parallelepipedal shape obtained considering only the part with $z \in [-h, \phi]$; the boundary surface of $R(\phi)$ shall be called $S_R(\phi)$ and the top surface of $S_R(\phi)$ (points $(x, y, z = \phi)$) shall be $A(\phi)$.

The mechanical energy in $R(\zeta)$ is now written subtracting from it the energy in the absence of the wave (when the region occupied by the fluid at rest is just $R(\phi)$):

$$\bar{E}_m(R(\zeta)) = \int_{R(\zeta)} \rho \left(\frac{1}{2} v^2 + g z \right) d^3x - \int_{R(\phi)} \rho g z d^3x$$

In the first term, the region above $z = \phi$ contributes an amount of energy that is approximated (neglecting higher-order infinitesimal terms) with the integrand value in $z = \phi$, then

integrating in x and y over $A(\phi)$ and multiplying by $\zeta(x,t)$; the potential energy contribution over the region $R(\phi)$ is also subtracted and we get

$$\bar{E}_m(R(\zeta)) = \int_{R(\phi)} \rho \frac{1}{2} v^2 d^3x + \int_{A(\phi)} \rho \frac{1}{2} v^2 \zeta(x,t) dx dy + \underbrace{\int_{A(\phi)} \left(\int_{z=\phi}^{z=\zeta} \rho g z dz \right) dx dy}_{\frac{1}{2} \rho g \zeta^2(x,t)} ;$$

once again the second term is not only negligible as a higher-order infinitesimal, but made of oscillating functions whose overall result is a vanishing time average, so

$$\bar{E}_m(R(\zeta)) = \frac{1}{2} \rho \int_{R(\phi)} v^2 d^3x + \frac{1}{2} \rho g \int_{A(\phi)} \zeta^2(x,t) dx dy$$

We shall now use some algebra to manipulate the first integral; indeed $v^2 = |\text{grad}\phi|^2$ and we can notice that $\text{div}(\phi \text{grad}\phi) = \text{grad}\phi \cdot \text{grad}\phi + \phi \nabla^2 \phi = |\text{grad}\phi|^2 = v^2$

$$\begin{aligned} \Rightarrow \bar{E}_m(R(\zeta)) &= \frac{1}{2} \rho \int_{R(\phi)} \text{div}(\phi \text{grad}\phi) d^3x + \frac{1}{2} \rho g \int_{A(\phi)} \zeta^2(x,t) dx dy = \\ &= \frac{1}{2} \rho \int_{S(R(\phi))} \phi \text{grad}\phi \cdot \vec{n} da + \frac{1}{2} \rho g \int_{A(\phi)} \zeta^2 dx dy \end{aligned}$$

We can cut out some parts of the flux integral over $S(R(\phi))$, and precisely where $\vec{v} = \text{grad}\phi$ has a zero normal component with respect to the integration surface, or where the net flux is zero:

- ⊙ At the bottom, $z = -h$, the b.c. requires $v_z = \text{grad}\phi \cdot \hat{e}_z = \phi$;
- ⊙ Due to y -invariance, $v_y = \text{constant}$ and thus on the two surfaces $y = y_0$, $y = y_0 + L$ of the volume $R(\phi)$ the incoming and outgoing fluxes are equal and balance out;
- ⊙ Periodicity along x requires incoming and outgoing fluxes through the surfaces at $x = x_0$, $x = x_0 + L$ to be equal and cancel out again.

What is left is just the flux through $A(\phi)$, in the vertical direction; so

$$\bar{E}_m(R(\zeta)) = \frac{1}{2} \rho \int_{A(\phi)} \phi(x, \phi, t) \frac{\partial \phi}{\partial z} \Big|_{z=\phi} dx dy + \frac{1}{2} \rho g \int_{A(\phi)} \zeta^2(x,t) dx dy =$$

recalling $\zeta = - \frac{1}{g} \frac{\partial \phi}{\partial t} \Big|_{z=\phi}$

$$= \frac{1}{2} \rho \int_{A(\varphi)} \left[\varphi(x, \varphi, t) \frac{\partial \varphi}{\partial z} \Big|_{z=\varphi} + \frac{1}{g} \left(\frac{\partial \varphi}{\partial t} \right)^2 \Big|_{z=\varphi} \right] dx dy$$

Let us rewrite explicitly φ and its derivatives before we perform the time average:

$$\varphi(x, z, t) = A \cosh(k(z+h)) \cos(kx - \omega t) \rightarrow \varphi(x, \varphi, t) = A \cosh(kh) \cos(kx - \omega t)$$

$$\frac{\partial \varphi}{\partial z} \Big|_{z=\varphi} = A k \sinh(kh) \cos(kx - \omega t)$$

$$\frac{\partial \varphi}{\partial t} \Big|_{z=\varphi} = A \omega \cosh(kh) \sin(kx - \omega t)$$

$$\Rightarrow \textcircled{1} \frac{1}{T} \int_T \varphi(x, \varphi, t) \frac{\partial \varphi}{\partial z} \Big|_{z=\varphi} dt = \frac{1}{T} \int_T \underbrace{A^2 k \sinh(kh) \cosh(kh)}_{\omega = \frac{1}{2} \sinh(2kh)} \cos^2(kx - \omega t) dt = \frac{1}{4} A^2 k \sinh(2kh)$$

$$\textcircled{2} \frac{1}{T} \int_T \frac{1}{g} A^2 \omega^2 \cosh^2(kh) \sin^2(kx - \omega t) dt = \frac{1}{g} A^2 \omega^2 \cosh^2(kh) = \text{using } \omega^2 = gk \tanh(kh)$$

$$= \frac{1}{2} A^2 k \sinh(kh) \cosh(kh) = \frac{1}{4} A^2 k \sinh(2kh)$$

We find here that the two terms are equal. This should not really come as a surprise: It is a more general fact that for small oscillations, the time averages of kinetic and potential energy are the same. So we finally end up writing

$$\langle \bar{E}_m(R(\zeta)) \rangle = \frac{1}{2} \rho \int_{A(\varphi)} \frac{1}{2} A^2 k \sinh(2kh) dx dy = \frac{1}{4} \rho A^2 k \sinh(2kh) \cdot \lambda L \quad \text{with } \lambda = 2\pi/k$$

$$\Rightarrow \langle \bar{E}_m(R(\zeta)) \rangle = \frac{1}{2} \bar{u} \rho L A^2 \sinh(2kh)$$

In general terms, we can write the flux of energy multiplying the energy density per unit volume $\rho \bar{e}_m$ times the flux velocity v_f and the area Σ of the cross section S through which we calculate the flux; inverting the expression, we can get v_f as follows:

$$\langle \dot{Q}^E(S) \rangle = \rho \bar{e}_m \Sigma v_f = \rho \bar{e}_m \frac{(\Sigma \lambda)}{\lambda} v_f = \frac{\langle \bar{E}_m(R(\zeta)) \rangle}{\lambda} v_f$$

$$\Rightarrow v_f = \lambda \langle \dot{Q}^E(S) \rangle / \langle \bar{E}_m(R(\zeta)) \rangle = \lambda \frac{1}{8} \rho L \omega A^2 [2kh + \sinh(2kh)] / \frac{1}{2} \bar{u} \rho L A^2 \sinh(2kh) =$$

$$= \frac{\lambda \omega}{2 \bar{u}} \left[1 + \frac{2kh}{\sinh(2kh)} \right] = \frac{\omega}{ek} \left[1 + \frac{2kh}{\sinh(2kh)} \right] = v_g$$

$v_f = v_g$ group velocity tells us that energy is not propagated at phase velocity, but at group velocity (even for a monochromatic wave*). Thus for gravity waves the deepest physical meaning of the group velocity is that of mean energy transport velocity (even more than that of translation velocity of a wave packet, which is always undergoing a progressive deformation in a dispersive medium).

[* = The carrier wavelength will be that of the wave itself.]

Appendix - Phase and group velocity (a very brief summary)

A monochromatic (single-wavelength) wave featuring an angular frequency ω and wavenumber k propagates with a velocity called PHASE VELOCITY $v_{ph} = \omega/k$ (with $k = 2\pi/\lambda$, λ wavelength). This ω/k ratio does not necessarily have to be constant; on the contrary, v_{ph} and ω can be a function of k . The expression $\omega(k)/k$ is called DISPERSION RELATION (the name comes from the fact that propagation in a "dispersive medium" will yield different velocities for waves with different wavelength/wavenumber, hence a "spreading" of a wavepacket made out of waves with a range of k values).

We can say that the dispersion relation is

NORMAL: v_{ph} is an increasing function of increasing λ

ANOMALOUS: v_{ph} is a decreasing function of increasing λ

TRIVIAL: $v_{ph} = \omega/k = \text{constant}$ (ω is a constant; see, e.g., d'Alembert's wave equation, for electromagnetic waves in vacuum)

Whatever dispersion relation we have, a monochromatic wave with angular frequency ω and wavenumber k can be expressed as

$$f(x, t) = A \exp[i(kx - \omega t)] = A \exp[ik(x - v_{ph}t)] = g(x - v_{ph}t)$$

for a simple, one-dimensional case; a more general solution to a wave equation is a whole superposition of monochromatic waves; in a continuous limit, this is an integral

$$F(x, t) = \int_{-\infty}^{+\infty} f(k) \exp[i(kx - \omega(k)t)] dk \quad \text{i.e. a wavepacket}$$

with $f(k)$ amplitude (a function that must be such that the integral is acceptable). A physically meaningful and interesting case is a superposition of waves within a limited range of k , i.e. $f(k) \neq 0$ outside a certain interval of k values and a peak amplitude within that range corresponding to a value $k = k_0$. The wave corresponding to such k_0 has maximum amplitude and is called CARRIER WAVE of the wavepacket. We now want to evaluate the propagation velocity of the wavepacket as a whole.

Since $f(k) \neq 0$ only over a limited k range, $\omega(k)$ can be expanded to first order:

$$\omega(k) \approx \omega(k_0) + \left. \frac{d\omega}{dk} \right|_{k=k_0} (k - k_0) = \omega(k_0) + \omega'_0 (k - k_0)$$

where we called $\omega'_0 = \left. \frac{d\omega}{dk} \right|_{k=k_0}$

$$\Rightarrow F(x,t) = \int_{-\infty}^{\infty} f(k) \exp[i(kx - \omega(k)t)] dk = \int_{k_1}^{k_2} f(k) \exp[i(kx - \omega(k)t)] dk$$

(where $[k_1, k_2]$ interval / $f(k) = \phi \forall k \notin [k_1, k_2]$; $k_0 \in [k_1, k_2]$; $f(k_0) = \max(f(k))$)

$$= \int_{k_1}^{k_2} f(k) \exp\left\{i \left[\underbrace{kx - (\omega_0 + (k-k_0)\omega'_0)t}_{kx - (k_0x + k_0t) - (k-k_0)\omega'_0 t} \right]\right\} dk =$$

$$kx - (k_0x + k_0t) - (k-k_0)\omega'_0 t = (k-k_0)(x - \omega'_0 t) + k_0x - \omega_0 t$$

$$= \int_{k_1}^{k_2} f(k) \exp\left\{i \left[(k-k_0)(x - \omega'_0 t) + \underbrace{k_0x - \omega_0 t}_{\text{a constant with respect to } k} \right]\right\} dk =$$

$$= \exp[i(k_0x - \omega_0 t)] \int_{k_1}^{k_2} f(k) \exp\left\{i \left[(k-k_0)(x - \omega'_0 t) \right]\right\} dk = \quad \text{with } \eta = k - k_0$$

$$= \exp[i(k_0x - \omega_0 t)] \int_{k_1 - k_0}^{k_2 - k_0} f(\eta + k_0) \exp[i\eta(x - \omega'_0 t)] d\eta =$$

$$= \exp[i(k_0x - \omega_0 t)] g(x - \omega'_0 t)$$

which represents a carrier wave $\exp[i(k_0x - \omega_0 t)]$ carrying the wave packet with an amplitude $g(x - \omega'_0 t)$; the translation velocity of this amplitude is ω'_0 , and we call it

$$\boxed{v_g = \omega'_0 = \left. \frac{d\omega}{dk} \right|_{k=k_0}} \quad \text{GROUP VELOCITY}$$

The propagation is not "painless": If in the expansion we had considered higher-order terms, in the calculation we would have found a slow deformation (dispersion) of the wavepacket. In other words, the wavepacket does not simply undergo a rigid translation.

The occurrence of a normal/anomalous dispersion relation impacts the value of v_g with respect to that of v_{ph} :

$$\frac{dv_{ph}}{dk} = \frac{d}{dk} \left(\frac{\omega}{k} \right) = \frac{1}{k^2} \left(\frac{d\omega}{dk} k - \omega \right) = \frac{1}{k^2} (v_g k - \omega) \gtrless \phi$$

$$\Rightarrow v_g k - \omega \gtrless \phi, \quad \text{i.e. } v_g \gtrless \omega/k = v_{ph}$$

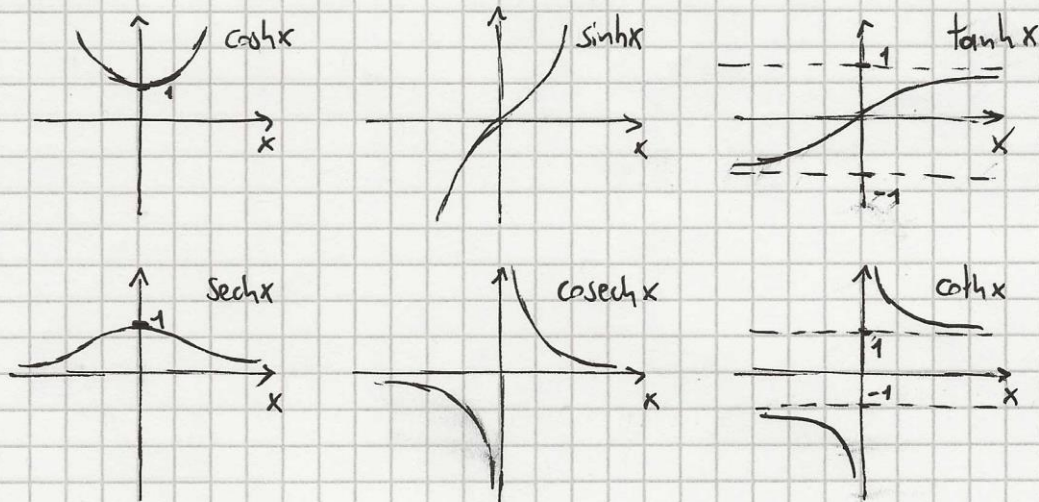
$$\text{So } v_g < v_{ph} \quad \Leftrightarrow \quad \frac{dv_{ph}}{dk} < \phi \quad \text{i.e. } \frac{dv_{ph}}{d\lambda} > \phi \quad \text{normal dispersion relation}$$

$$v_g > v_{ph} \quad \Leftrightarrow \quad \frac{dv_{ph}}{dk} > \phi \quad \text{i.e. } \frac{dv_{ph}}{d\lambda} < \phi \quad \text{anomalous dispersion relation}$$

Appendix - A digest of hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}; \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} x = 1/\cosh x; \quad \operatorname{cosech} x = 1/\sinh x \quad \forall x \neq 0; \quad \operatorname{coth} x = 1/\tanh x \quad \forall x \neq 0$$

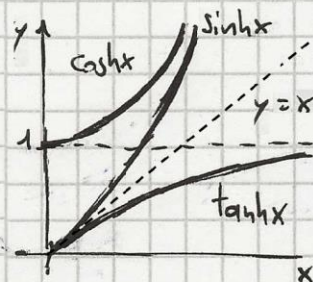


Properties

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx}(\cosh x) = \sinh x; \quad \frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$



Double angle formulae: $\cosh(2x) = \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

$$\sinh(2x) = 2\sinh x \cosh x$$

$$\tanh(2x) = 2\tanh x / (1 + \tanh^2 x)$$

Taylor expansion for $x \rightarrow 0$

$$\sinh x \approx x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\tanh x \approx x - \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

$$\operatorname{coth} x \approx \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \dots$$

Integrals

$$\int \sinh^2(ax) dx = \frac{1}{4a} \sinh(2ax) - \frac{x}{2} + C$$

$$\int \cosh^2(ax) dx = \frac{1}{4a} \sinh(2ax) + \frac{x}{2} + C$$