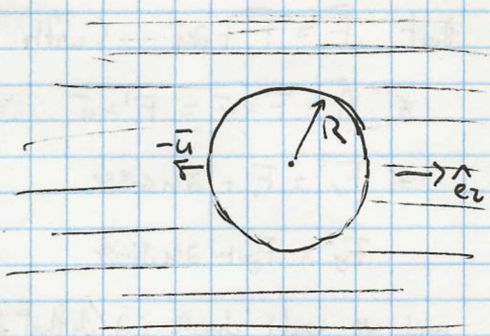
Stokes flow for a solid sphere in a viscous fluid

Let us consider a sphere of radius R within a large (infinite) domain filled with an incompressible viscous fluid. The fluid is at rest far from the sphere, while the latter is moving with a constant velocity $-\bar{u}$ to the left. The problem is equivalent to a sphere dragged by a fluid moving with velocity $\bar{u}$ to the right (and otherwise at rest), simply by a change of c.s.



We address the problem a) in a steady-state condition, b) for the viscosity-dominated case $Re \to \phi$; this means we will solve the Stokes equation

$$-\text{grad}\, p + \eta \nabla^2 \bar{v} = \phi \qquad \text{with } \bar{v} = \bar{u} \text{ at infinity (b.c. for the fluid velocity field).}$$

⊙ In a cs where the sphere moves, we have $\bar{v}' = \bar{v} - \bar{u}$ fluid velocity (with $\bar{v}' = \phi$ at infinity)

since the flow is incompressible, $\text{div}(\bar{v}') = \text{div}(\bar{v} - \bar{u}) = \text{div}(\bar{v}) - \cancel{\text{div}(\bar{u})}^{\text{uniform}} = \phi$

and therefore $\exists \bar{A} \,/\, \bar{v}' = \text{curl}\, \bar{A}$, with $\lim_{\bar{r} \to \infty} \text{curl}\, \bar{A} = \phi$ ($\bar{A}$ is a vector field that must be irrotational very far from the sphere)

⊙ The problem features an apparent cylindrical symmetry with respect to the $\hat{e}_z$ axis, identified as the sphere's velocity direction, i.e. there is a φ-invariance (with $\hat{e}_\varphi$ the azimuthal angle direction). Hence we look for a solution featuring the same symmetry, i.e. a $\bar{v}'$ independent of φ and without v_φ component: $\underline{\bar{v} = \bar{v}'(r,\vartheta) = \text{curl}\, \bar{A} = f(r,\vartheta) \text{ with } (\text{curl}\, \bar{A})_\varphi = \phi}$; as a consequence,
$\underline{\underline{(\text{curl}\, \bar{A})_\varphi = \phi \iff \bar{A} = A\hat{e}_\varphi \Rightarrow \bar{A}\cdot\bar{u} = \phi}}$

⊙ We can assume $\bar{A}$ to depend linearly on $\bar{u}$, since the problem is defined by a linear eq. in $\bar{u}$ with b.c. also linear in $\bar{u}$; since $\bar{A}\cdot\bar{u} = \phi$, $\bar{A}$ must have a structure like
$\underline{\bar{A} = \bar{F} \times \bar{u}}$ with $\bar{F} = \bar{F}(r,\vartheta)$ but independent of $\bar{u}$ ($\Rightarrow \bar{A} \propto \bar{u}$ and $\bar{A} \perp \bar{u}$)

⊙ Since $\bar{A} \parallel \hat{e}_\varphi$ and $\bar{F} \perp \bar{A}$, $\Rightarrow$ the components of $\bar{F}$ can only be F_r, F_ϑ. But the features of the dynamics contained in the problem's solution $\bar{A}$ must derive from $\bar{u}$, we make the hypothesis that $\bar{F}$ is accounting only for the geometry only of the problem, i.e. specifically the geometry of the solid body. As the latter is a sphere, by spherical symmetry we conclude that $\bar{F}$ is purely radial, i.e. a spherically symmetric vector function directed along $\hat{e}_r$ (with origin set in the sphere's center): $\bar{F} = f'(\bar{r})\hat{e}_r$.

We can find a more solid justification to this hypothesis of spherical invariance and in particular to the radial orientation of $\bar{F}$. If this were not the case, we could have $\bar{F}'$ such that $\bar{F}' = \bar{F} + a\bar{u}$ (with $a \in \mathbb{R}$ arbitrary constant) $\Rightarrow$

* $\bar{A} = \bar{F} \times \bar{u} = \bar{F}' \times \bar{u}$ $\Rightarrow$ in the end this would not change the solution $\bar{A}$
* $F_r' = F_r + au\cos\theta$

  $F_\theta' = F_\theta - au\sin\theta$

and we could choose a such that $a = F_\theta / u\sin\theta$ $\Rightarrow F_\theta' = \phi$

i.e. starting from a non-radial $\bar{F}$ we could always built a radial $\bar{F}'$ and obtain the same solution $\bar{A}$. Hence the hypothesis is acceptable – and anyway verified as such *a posteriori* since it is a solution to the Stokes equation.

⊙ Summarizing, we have $\bar{A} = \bar{F} \times \bar{u} = f'(r)\hat{e}_r \times \bar{u} = -uf'(r)\sin\theta\,\hat{e}_\varphi = A_\varphi \hat{e}_\varphi$

and $\bar{v}' = \text{curl}\,\bar{A} = \text{curl}\,(f'(r)\hat{e}_r \times \bar{u})$

The function $f'(r)\hat{e}_r$ can be written as the gradient of a scalar function $f(r)$,

$f'(r)\hat{e}_r = \text{grad}\,(f(r)) \Rightarrow$

$$\bar{A} = \bar{F} \times \bar{u} = \text{grad}(f(r)) \times \bar{u} = \text{curl}(f(r)\bar{u}) \quad \Leftarrow \left(\text{curl}(f\bar{u}) = f\,\underbrace{\text{curl}\,\bar{u}}_{\phi} + \text{grad}\,f \times \bar{u}\right)$$

and $\bar{v}' = \text{curl}\,\bar{A} = \text{curl}(\text{curl}\,(f\bar{u}))$

⊙ We come back now to the Stokes eq., but in its equivalent form

$$\Delta(\text{curl}\,\bar{v}) = \phi \;^{*} \qquad \text{that implies} \qquad \Delta(\text{curl}\,\bar{v}') = \phi$$

Let us work us this expression:

$\Delta(\text{curl}\,\bar{v}') = \Delta\left\{\text{curl}[\text{curl}(\text{curl}(f\bar{u}))]\right\} =$ using $\text{curl}(\text{curl}\,\bar{g}) = \text{grad}(\text{div}\,\bar{g}) - \Delta\bar{g}$, with $\bar{g} = \text{curl}(f\bar{u})$

$$= \Delta\left\{\text{grad}\,[\underbrace{\text{div}(\text{curl}(f\bar{u}))}_{\text{div}(\text{curl}) = \phi}] - \Delta(\text{curl}(f\bar{u}))\right\} = -\Delta^2[\text{curl}(f\bar{u})] = \phi$$

Since $\text{curl}(f\bar{u}) = \text{grad}\,f \times \bar{u}$, $\Rightarrow \Delta^2[\text{curl}(f\bar{u})] = \Delta^2(\text{grad}\,f \times \bar{u}) \underset{\uparrow}{=} \Delta^2(\text{grad}\,f) \times \bar{u} = \phi$

since $\bar{u}$ constant vector

$\Delta^2(\text{grad}\,f) \times \bar{u} = \phi \Leftrightarrow \Delta^2(\text{grad}\,f) \parallel \bar{u}$, i.e. $\underline{\Delta^2(\text{grad}\,f) = a\bar{u}}$

Now let us recall the fact that $f = f(r)$ only, $\Rightarrow \Delta^2(\text{grad}\,f)$ = function of r only too; the radial component of $\Delta^2(\text{grad}\,f)$ is $\Delta^2 \dfrac{\partial f(r)}{\partial r} = au\cos\theta$ that would depend on θ, $\Rightarrow$

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* = We indicate the Laplacian operator as Δ instead of ∇^2 to avoid confusion in the following calculations.

$\Rightarrow$ it is mandatory that $a = \phi$, so $\underline{\Delta^2(\text{grad}\, f) = \phi}$.

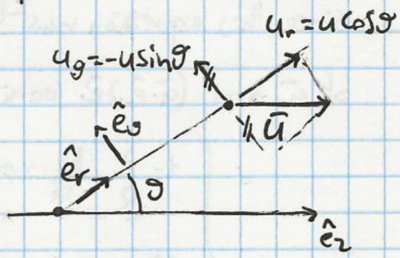
Integrating once this equation we get

$$\Delta^2 f = \text{constant},$$

where the constant must be zero to ensure the b.c. $\bar v' = \phi$ at infinity (derivatives of f must vanish so that $\bar v'$ derivatives do, too as required by smoothly vanishing $\bar v'$ at infinity).

Purely radial dependence of f reduces $\Delta^2 f = \phi$ to

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\right]^2 f(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right)\right] = \phi$$



and it can be proved that solutions take the form *

$f(r) = ar + b/r$ which we can finally insert in $\bar v$:

$$\bar v = \bar v' + \bar u = \bar u + \text{curl}\,\bar A = \bar u + \text{curl}[\text{curl}(f\bar u)]$$

With a certain amount of algebra, noting $f\bar u = (f\bar u)_r \hat e_r + (f\bar u)_\vartheta \hat e_\vartheta = fu\cos\vartheta\,\hat e_r - fu\sin\vartheta\,\hat e_\vartheta$,

$$\bar A = \text{curl}(f\bar u) = \hat e_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}[r(f\bar u)_\vartheta] - \frac{\partial}{\partial\vartheta}(f\bar u)_r\right] = \dots = \left[\frac{b}{r^2} - a\right] u\sin\vartheta\,\hat e_\varphi = A_\varphi \hat e_\varphi$$

$$\bar v' = \text{curl}\,\bar A = \text{curl}(A_\varphi \hat e_\varphi) = \hat e_r \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(A_\varphi \sin\vartheta)\right] + \hat e_\vartheta\left[-\frac{\partial}{\partial r}(rA_\varphi)\right] =$$

$$= -a\,\frac{2u\cos\vartheta\,\hat e_r - u\sin\vartheta\,\hat e_\vartheta}{r} + b\,\frac{2u\cos\vartheta\,\hat e_r + u\sin\vartheta\,\hat e_\vartheta}{r^3} =$$

$$= -a\,\frac{u\cos\vartheta\,\hat e_r + \overbrace{u\cos\vartheta\,\hat e_r - u\sin\vartheta\,\hat e_\vartheta}^{\bar u}}{r} + b\,\frac{3u\cos\vartheta\,\hat e_r \overbrace{- u\cos\vartheta\,\hat e_r + u\sin\vartheta\,\hat e_\vartheta}^{-\bar u}}{r^3} =$$

$$= -a\,\frac{\bar u + (\bar u\cdot\hat e_r)\hat e_r}{r} + b\,\frac{3(\bar u\cdot\hat e_r)\hat e_r - \bar u}{r^3}$$

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* = Let us prove it explicitly; we have $\Delta^2 f = \phi$, and by calling $g \doteq \Delta f$, $\Rightarrow \Delta g = \phi$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg(r)}{dr}\right) = \phi \rightsquigarrow \frac{d}{dr}\left(r^2\frac{dg}{dr}\right) = \phi \rightarrow r^2\frac{dg}{dr} = \alpha \rightsquigarrow g = -\frac{\alpha}{r} + \gamma$$ but $\gamma = \phi$ to satisfy zero b.c.

Now $\Delta f = g \rightsquigarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = -\frac{\alpha}{r} \rightsquigarrow \frac{d}{dr}\left(r^2\frac{df}{dr}\right) = -\alpha r \rightsquigarrow r^2\frac{df}{dr} = -\frac{\alpha}{2}r^2 + \beta \rightsquigarrow \frac{df}{dr} = -\alpha + \frac{\beta}{r^2} \rightsquigarrow$

$f(r) = -\frac{\alpha}{2}r - \frac{\beta}{r} + \delta$ with $\delta = \phi$ without loss of generality, since $\bar v'$ depends on derivatives of f;

so by renaming $a \doteq -\alpha/2$ $b = -\beta$ we get $\underline{f(r) = ar + b/r}$

Thus we have $\bar{v} = \bar{u} + \bar{v}' = \bar{u} - a\,\dfrac{\bar{u} + (\bar{u}\cdot\hat{e}_r)\hat{e}_r}{r} + b\,\dfrac{3(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}}{r^3}$

with the constants a, b to be found using the b.c.; since $\bar{v}(r=R, \vartheta) = \phi$, i.e. no slip on the surface of the static sphere within the flowing fluid,

$$\bar{u} - \frac{a}{R}\bar{u} - \frac{a(\bar{u}\cdot\hat{e}_r)\hat{e}_r}{R} + \frac{3b(\bar{u}\cdot\hat{e}_r)\hat{e}_r}{R^3} - \frac{b\bar{u}}{R^3} = \phi \quad \text{and rearranging,}$$

$$\bar{u}\left(1 - \frac{a}{R} - \frac{b}{R^3}\right) + (\bar{u}\cdot\hat{e}_r)\hat{e}_r\left(\frac{3b}{R^3} - \frac{a}{R}\right) = \phi \ ;$$

Since this expression must hold for any radial direction $\hat{e}_r$, it is satisfied by having the coefficients of $\bar{u}$ and $(\bar{u}\cdot\hat{e}_r)\hat{e}_r$ vanish separately;

$$\begin{cases} 1 - \dfrac{a}{R} - \dfrac{b}{R^3} = \phi \\ \dfrac{3b}{R^3} - \dfrac{a}{R} = \phi \end{cases} \quad \text{that is solved with } \underline{\underline{a = \frac{3}{4}R}},\ \underline{\underline{b = \frac{1}{4}R^3}}$$

so that we finally obtain

$$f(r) = \frac{3R}{4}r + \frac{1}{4}\frac{R^3}{r} \quad \text{and fluid velocity around a fixed sphere is}$$

$$\boxed{\bar{v}(r, \vartheta) = \bar{u} - \frac{3R}{4}\,\frac{\bar{u} + (\bar{u}\cdot\hat{e}_r)\hat{e}_r}{r} + \frac{R^3}{4}\,\frac{3(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}}{r^3}}$$

or, in components,

$$v_r = \bar{v}\cdot\hat{e}_r = u\cos\vartheta\left[1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right]$$

$$v_\vartheta = \bar{v}\cdot\hat{e}_\vartheta = -u\sin\vartheta\left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right]$$

A few notable remarks about this solution.

- Comparison with a sphere in an ideal fluid (potential flow)

The solution we found for a sphere moving with velocity $\bar{u}$ to the right within a fluid at rest for $r \to \infty$ was a fluid velocity fluid $\bar{v}'_{pot}$

$$\bar{v}'_{pot} = \frac{R^3}{2}\,\frac{3(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}}{r^3}$$

Now let us flip left and right (change of sign) and change cs to a static sphere within a fluid with velocity $\bar{u}$ to the right at $r \to \infty$ (add $+\bar{u}$) $\Rightarrow$

$$\bar{v}_{pot} = \bar{u} - \frac{R^3}{2}\,\frac{3(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}}{r^3}$$

and let us compare it to the viscous velocity field $\bar{v}_{visc}$ we just found,

$$\bar{v}_{visc} = \bar{u} - \frac{3R}{4}\,\frac{\bar{u} + (\bar{u}\cdot\hat{e}_r)\hat{e}_r}{r} + \frac{R^3}{4}\,\frac{3(\bar{u}\cdot\hat{e}_r)\hat{e}_r - \bar{u}}{r^3}$$

Notice that the last term has the same structure both in $\bar{v}_{pot}$ and $\bar{v}_{visc}$, apart from a factor $-1/2$; it is a $\sim \frac{1}{r^3}$ irrotational term (a "dipole" term), and indeed potential flow is irrotational by its very definition; in the Stokes problem there is an added, rotational term $\sim 1/r$ (curl vanishing as $1/r^2$ at infinity).

◆ Pressure field

$$\mathrm{grad}\,p = \eta\Delta\bar{v} = \eta\Delta\bar{v}' = \eta\,\Delta\left[\mathrm{curl}(\mathrm{curl}(f\bar{u}))\right] = \eta\,\Delta\left[\mathrm{grad}(\mathrm{div}(f\bar{u})) - \bar{u}\Delta f\right] =$$

$$= \mathrm{grad}\left[\eta\Delta\underbrace{(\mathrm{div}(f\bar{u}))}_{\bar{u}\cdot\mathrm{grad}f + \cancel{f\,\mathrm{div}\bar{u}}}\right] - \eta\underbrace{\Delta(\bar{u}\Delta f)}_{\bar{u}\Delta^2 f = 0} = \mathrm{grad}\left[\eta\Delta(\bar{u}\cdot\mathrm{grad}f)\right] =$$

$$= \mathrm{grad}\left[\eta\bar{u}\cdot\Delta(\mathrm{grad}f)\right] = \mathrm{grad}\left[\eta\bar{u}\cdot\mathrm{grad}(\Delta f)\right]$$

Now since $\mathrm{grad}\,p = \mathrm{grad}\left[\eta\bar{u}\cdot\mathrm{grad}(\Delta f)\right] \Rightarrow p = p_0 + \eta\bar{u}\cdot\mathrm{grad}(\Delta f)$

and using $f(r) = \frac{3R}{4}r + \frac{R^3}{4r}$ we have $\Delta f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = \ldots = \frac{3R}{2r}$

so $\mathrm{grad}(\Delta f) = \hat{e}_r\frac{d}{dr}(\Delta f) = -\frac{3R}{2r^2}\hat{e}_r$

and finally $p(r) = p_0 - \frac{3}{2}\eta R\frac{\bar{u}\cdot\hat{e}_r}{r^2}$; notice that $\lim_{r\to\infty} p(r) = p_0$ the pressure in the fluid without the perturbing obstacle

◆ Force exerted on the sphere

The force exerted by the moving fluid onto the sphere, or conversely the resistance force exerted by the fluid onto the moving sphere, can be calculated considering at first the force on the infinitesimal surface element da of the sphere;

$\sigma_{ij}n_j\,da$, or, in spherical coordinates (with polar axis $\parallel \bar{u} \Rightarrow \hat{e}_z$)

$\sigma_{rj}n_j R^2\sin\theta\,d\theta\,d\varphi$

and we expect a force along $\bar{u}$ (and φ-invariant); with $\sigma_{ij} = \sigma'_{ij} - p\delta_{ij}$ and taking the projection onto the $\bar{u}$ direction ($n_j = \hat{e}_z$) of all components of the force (both normal and tangential to the surface)

$\sigma_{r_j} n_j = -p\cos\vartheta + \sigma'_{rr}\cos\vartheta - \sigma'_{r\vartheta}\sin\vartheta$

we can perform the integral over the whole surface and get the resultant force F along $\hat{e}_z$:

$$F = \int_\phi^{2\pi}\int_\phi^{\pi} \left(-p\cos\vartheta + \sigma'_{rr}\cos\vartheta - \sigma'_{r\vartheta}\sin\vartheta\right) R^2 \sin\vartheta \, d\vartheta \, d\varphi$$

where

$$p(R) = p_0 - \frac{3}{2}\frac{\eta u\cos\vartheta}{R} \; ; \quad \sigma'_{rr}(R) = 2\eta \frac{\partial v_r}{\partial r}\bigg|_{r=R} = 2\eta u\cos\vartheta \left[\frac{3R}{2r^2} - \frac{3R^3}{2r^4}\right]_{r=R} = \phi \; ;$$

$$\sigma'_{r\vartheta}(R) = \eta\left[\frac{1}{r}\frac{\partial v_r}{\partial\vartheta} + \frac{\partial v_\vartheta}{\partial r} - \frac{v_\vartheta}{r}\right]\bigg|_{r=R} = \eta\left\{-u\sin\vartheta\left[\cancel{\frac{1}{r} - \frac{3R}{4r} + \frac{R^3}{2r^3}}\right] - u\sin\vartheta\left[\frac{3R}{4r^2} + \frac{3R}{4r^4}\right] + u\sin\vartheta\left[\cancel{\frac{1}{r} - \frac{3R}{4r} + \frac{R^3}{2r^3}}\right]\right\}\bigg|_{r=R} =$$

$$= -\eta u\sin\vartheta\left(\frac{3}{4R} + \frac{3}{4R}\right) = -\frac{3\eta}{2R}u\sin\vartheta$$

$$\Rightarrow F = \int_\phi^{2\pi}\int_\phi^{\pi}\Big[-p_0\cos\vartheta + \underbrace{\frac{3\eta}{2R}u\cos^2\vartheta + \frac{3\eta}{2R}u\sin^2\vartheta}_{3\eta u/2R}\Big] R^2\sin\vartheta\, d\vartheta\, d\varphi =$$

$$\int_\phi^{\pi}\cos\vartheta\sin\vartheta\, d\vartheta = \int_\phi^{\pi}\sin(2\vartheta)\,d\vartheta = \phi$$

$$= \frac{3\eta u R}{2}\int_\phi^{2\pi}\int_\phi^{\pi}\sin\vartheta\, d\vartheta\, d\varphi = \underline{\underline{6\eta\pi u R \quad \text{Stokes' law}}} \quad \text{for the drag on the sphere}$$

Notice that F is linear in u and R. This fact could be deduced without an explicit calculation, just by means of dimensional analysis: Density ρ does not appear in the Stokes equation, hence the force must depend only on η, u, R and the only combination of these quantities yielding something with the dimensions of a force is indeed the linear combination $\eta u R$ (once again, thank Landau for this smart observation).

The force on a slowly moving object ($Re \to \phi$) with a non-spherical, arbitrary shape shall be more complicated, although qualitatively similar. The direction of the drag force will no be parallel to the body's velocity, and the relationship between $\vec{F}$ and $\vec{u}$ will be written as

$$F_i = \eta\, a_{ik} u_k$$

with a_{ik} a 2nd-order tensor independent of $\vec{u}$ and containing the information about the problem's geometry.

* Refinement of Stokes' formula — Oseen's formula and penetration coefficient

Stokes' eq. is unsatisfactory at large distances from the sphere, even at low Re, because the advective term becomes non-negligible anyway. Indeed $\lim_{\bar{r}\to\infty} \bar{v} = \bar{u}$, and the dominant term in $\bar{v}$, for Stokes' approximation, is $\sim uR/r$

$\Rightarrow \bar{v}\cdot \text{grad}\,\bar{v} \sim u \cdot \text{grad}\,(uR/r) \sim u^2R/r^2$ for $r \to \infty$;

the viscous term $\nu\nabla^2\bar{v}$ contains second derivatives of $\bar{v}$ $\Rightarrow \nu\nabla^2\bar{v} \underset{r\to\infty}{\sim} \nu uR/r^3$

and requesting the advective term to be negligible

$$\frac{u^2R}{r^2} \ll \frac{\nu uR}{r^3} \quad \Rightarrow \underline{\underline{r \ll \nu/u}}$$

is equivalent to setting a limit on the distance from the sphere beyond which the Stokes eq. loses its validity.

An improvement comes from <u>Oseen's approximation</u>: the advective term is included in the equation, in the approximation $(\bar{v}\cdot\text{grad})\bar{v} \to (\bar{u}\cdot\text{grad})\bar{v}$ (better indeed at large r) so that

$$(\bar{u}\cdot\text{grad})\,\bar{v} = -\frac{1}{\rho}\,\text{grad}\,p + \nu\nabla^2\bar{v}$$

The result (which we shall not derive here) is that the drag force contains another term: Since we considered a term that is of higher order in $\bar{u}$, so is the force, which can be seen as an expansion in powers of the Reynolds number (equivalent to powers of u, if we fix all of the remaining parameters):

$$\bar{F} = 6\pi\eta\bar{u}R\,(1 + 3uR/8\nu)$$

The force can also be conveniently expressed using the <u>penetration coefficient</u>, a dimensionless quantity $C(Re)$ that accounts for the geometric features of the problem and is constant once the properties of motion (i.e., Re) are set. We define C through the expression

$$\bar{F} = \frac{1}{2}\rho U^2 S\, C(Re) \qquad \text{(notice } \rho U^2 S \text{ is dimensionally a force)}$$

where S is the area of a suitable surface (perpendicular to $\bar{F}$, and to $\bar{u}$ as well if $\bar{F}/\!/\bar{u}$); in the case of the sphere using $L = 2R$ to define Re, then $S = \pi R^2$, $u = U$ and

* Stokes' formula

$$F = 6\pi\eta\bar{u}R = \frac{1}{2}\rho u^2 \pi R^2 \frac{24\eta}{\rho u\cdot 2R} = \frac{1}{2}\rho u^2 S \frac{24}{Re}$$

$\Rightarrow C = 24/Re$ (experimentally verified as valid for $Re < 0.1$).

* Oseen's formula

$$F = 6\pi\eta\bar{u}R\left(1 + \frac{3uR}{8\nu}\right) = \frac{1}{2}\rho u^2 S \frac{24}{Re}\left(1 + \frac{3}{16}\frac{u\cdot 2R}{\nu}\right) = \frac{1}{2}\rho u^2 S \frac{24}{Re}\left(1 + \frac{3}{16}Re\right)$$

$$\Rightarrow \quad C = \frac{24}{Re}\left(1 + \frac{3}{16} Re\right) \qquad \text{(experimentally verified as valid for } Re < 0.8)$$

* An empirical formula was found to yield

$$C = \frac{24}{Re}\left(1 + \frac{3}{8} Re\right)^{1/2} \qquad \text{valid for } Re < 100$$