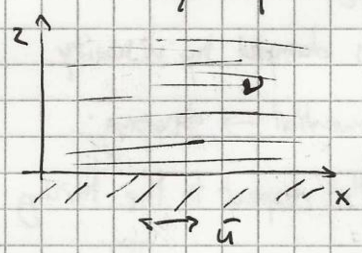
Oscillatory motions in viscous fluids

Infinitely deep basin with an oscillating bottom



We consider a basin of incompressible viscous fluid with infinite height on top of a solid bottom at $z=\phi$. The fluid is also unbounded in x and y.

The b.c. to the Navier-Stokes eq. is given by an oscillation of the bottom along $\hat{e}_x$, i.e. parallel to the plane itself: $\bar{u}(t) = u_0 e^{-i\omega t}\hat{e}_x$. Considering the x- and y-invariance of the problem and the b.c. velocity along $\hat{e}_x$ only, we look for a steady-state solution (i.e. a solution with a periodic dependence on time, setting in after the transient) obeying these symmetries, i.e.

$\bar{v}(z,t) = v_x(z,t)\,\hat{e}_x$ which satisfies incompressibility ($\mathrm{div}\,\bar{v}=\phi$)

Since $(\bar{v}\cdot\mathrm{grad})\bar{v} = v_x\cancel{\partial_x \bar{v}} + \cancel{v_y\partial_y \bar{v}} + \cancel{v_z}\partial_z \bar{v} = \phi$,

and $(\mathrm{grad}\,p)_x = \phi$ as no pressure force is exerted along $\hat{e}_x$ by external forces, we come to write the x-component of the Navier-Stokes eq.

$$\frac{\partial v_x}{\partial t} = \nu \nabla^2 v_x \quad \rightsquigarrow \quad \frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial z^2}$$

Since the b.c. is oscillatory, we guess a solution in the form

$v_x(z,t) = v_0 \exp[i(kz-\omega t)]$ and plugging it into the Navier-Stokes eq.,

$$\frac{\partial v_x}{\partial t} = -i\omega v_0 \exp[i(kz-\omega t)] = \nu\frac{\partial^2 v_x}{\partial z^2} = -\nu k^2 v_0 \exp[i(kz-\omega t)]$$

that is $\boxed{k^2 = i\omega/\nu}$ and since $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$

$$\underline{k = \pm\sqrt{\omega/2\nu}\,(1+i)}$$ dispersion relation with two possible values of k;

careful though, for if we take the one with the minus sign, in the exponential we get

$ikz = z\sqrt{\omega/2\nu}\,(-i)(1+i) = z\sqrt{\omega/2\nu}\,(1-i)$, i.e. $\mathrm{Re}(ikz) \geq \phi$

and we have an exponential with positive real exponent, i.e. $\lim_{z\to\infty} v_x \to \infty$ (unphysical, as as not only should the solution be finite, but also we are getting farther and farther from the perturbation and we expect the motion to decay).

Therefore we say there is only one acceptable wavenumber

$$k = \sqrt{\omega/2\nu}\,(1+i) = (1+i)/\delta$$

where we define $\delta \doteq \sqrt{2\nu/\omega}$ penetration depth, a length scale over which indeed we can see from the form of the solution that the perturbation is damped by viscosity.

The occurrence of an imaginary part in k yields a real exponential $\rightarrow$ damping;

The occurrence of a real part in k yields a phase shift with respect to the forcing:

$$i(kz-\omega t) = i\left[\frac{1}{\delta}(1+i)z - \omega t\right] = -\frac{z}{\delta} + i\left(\frac{z}{\delta} - \omega t\right)$$

$\frac{z}{\delta}$ $\hookrightarrow$ phase difference with respect to $u \sim e^{-i\omega t}$

$$\Rightarrow \quad v_x(z,t) = v_0\, e^{-z/\delta} \exp[i(z/\delta - \omega t)]$$

The no-slip b.c. requires $v_x(\phi,t) = v_0 e^{-i\omega t} = u_0 e^{-i\omega t} \Rightarrow v_0 = u_0$

so finally the complete solution is $\boxed{\bar{v}(z,t) = u_0 e^{-z/\delta} \exp[i(z/\delta - \omega t)]\,\hat{e}_x}$ $\left(= e^{-z/\delta} e^{iz/\delta} \bar{u}(t)\right)$

a transverse wave (motion and propagation direction are perpendicular) with a damping along the propagation direction $\hat{e}_z$ and a phase shift that depends periodically on the distance from the perturbation at $z = \phi$.

The shear stress (friction) per unit area on the solid bottom is

$$\sigma_{zx} = \eta \frac{\partial v_x}{\partial z}\Big|_{z=\phi} = \left\{ -\frac{\eta}{\delta} u_0 e^{-z/\delta} \exp[\dots] + i\frac{\eta}{\delta} u_0 e^{-z/\delta} \exp[\dots] \right\}\Big|_{z=\phi} =$$

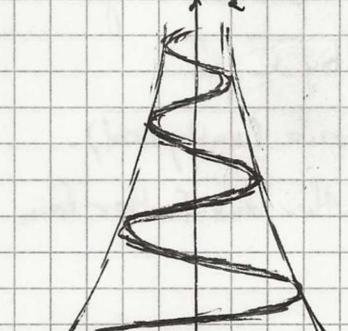
$$= \frac{\eta}{\delta}(i-1)\, u_0 e^{-i\omega t} = \eta \sqrt{\frac{\omega}{2\nu}}\,(i-1) u_0 e^{-i\omega t} = \sqrt{\frac{1}{2}\omega\rho\eta}\,(i-1)\, u(t) \qquad \left(\frac{1}{\sqrt{2}}(i-1) = e^{i3\pi/4}\right)$$

notice a phase difference between forcing and friction, too

The time average of power per unit area dissipated through friction is

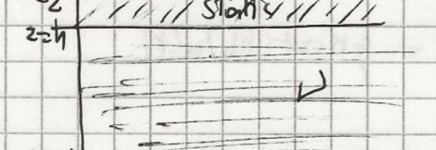
$$\langle \sigma_{zx} u(t) \rangle = \left\langle \mathrm{Re}\left[\sqrt{\frac{1}{2}\omega\rho\eta}\,(i-1)\, \underbrace{u^2(t)}\right] \right\rangle = -\frac{1}{2} u_0^2 \sqrt{\omega\rho\eta/2}$$

$$\langle u^2(t) \rangle = \frac{1}{2} u_0^2$$



peaks and troughs travel along $(-)\hat{e}_z$ but also get damped along $\hat{e}_z$ (much more strongly, or "faster in z", than depicted here)

Viscous fluid oscillation between two parallel plates



Here we consider again an incompressible viscous flow unbounded in x and y, but limited in z by two solid walls. The top one ($z=h$) is still, while the bottom one ($z=\phi$) is oscillating along $\hat{e}_x$ with velocity $\bar{u}(t) = u_0 e^{-i\omega t}\hat{e}_x$.

Once again we notice the x- and y-invariance of the system and we guess a solution $\bar{v}(z,t) = v_x(z,t)\hat{e}_x$ thus formulating the x-component of the problem as

$$\begin{cases} \dfrac{\partial v_x}{\partial t} = \nu \dfrac{\partial^2 v_x}{\partial z^2} & \text{x-component of the Navier-Stokes eq.} \\ \left.\begin{aligned} v_x(\phi,t) &= u(t) = u_0 e^{-i\omega t} \\ v_x(h,t) &= \phi \end{aligned}\right\} & \text{no-slip conditions at the walls} \end{cases}$$

with an explicit guess solution $\quad v_x(z,t) = [A\sin(\kappa z) + B\cos(\kappa z)]e^{-i\omega t}$ * ;

plugging it into the Navier-Stokes eq.,

$$-i\omega[A\sin(\kappa z) + B\cos(\kappa z)]e^{-i\omega t} = -\nu\kappa^2[A\sin(\kappa z) + B\cos(\kappa z)]e^{-i\omega t}$$

yielding the same dispersion relation found before

$\kappa^2 = i\omega/\nu \quad$ where we can take $\kappa = \sqrt{\omega/2\nu}\,(1+i)$ (one can see that the solution $\kappa = -\sqrt{\omega/2\nu}\,(1+i)$ yields the same final result)

and by defining $\delta \doteq \sqrt{2\nu/\omega} \Rightarrow \underline{\underline{\kappa = (1+i)/\delta}}$

With the b.c. we have

$$\odot \; v(\phi,t) = [A\sin(\phi) + B\cos(\phi)]e^{-i\omega t} = u_0 e^{-i\omega t} \Rightarrow \boxed{B = u_0}$$

$$\odot \; v(h,t) = (A\sin(\kappa h) + u_0\cos(\kappa h))e^{-i\omega t} = \phi \Rightarrow \boxed{A = -u_0\cot(\kappa h)}$$

So we can manipulate $\cos(\kappa z) - \cot(\kappa h)\sin(\kappa z) = \dfrac{\sin(\kappa h)\cos(\kappa z) - \cos(\kappa h)\sin(\kappa z)}{\sin(\kappa h)} =$

$$= \frac{\sin(\kappa h - \kappa z)}{\sin(\kappa h)} = \frac{\sin[\kappa(h-z)]}{\sin(\kappa h)}$$

$$\Rightarrow \bar{v}(z,t) = \frac{\sin[\kappa(h-z)]}{\sin(\kappa h)} u_0 e^{-i\omega t}\hat{e}_x = \frac{\sin[\kappa(h-z)]}{\sin(\kappa h)}\bar{u}(t)$$

(see here that $\kappa = \pm(1+i)\delta$ does not make a difference)

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* = That is the form proposed, e.g., by Landau and it is equivalent to a linear combination of waves propagating with opposite orientation along z. Just for the fun of it, let us see it on the next page →

The friction force per unit area on the two wall surfaces is:

in $z=\phi$ $\quad f_x^{(\phi)} = \sigma_{zx}\Big|_{z=\phi} = \eta \partial_z v_x\Big|_{z=\phi} = -\eta k \dfrac{\cos(kh)}{\sin(kh)}\, u(t) = -\eta k \cot(kh)\, u(t)$

in $z=h$ $\quad f_x^{(h)} = -\sigma_{zx}\Big|_{z=h} = -\eta \partial_z v_x\Big|_{z=h} = \eta k \dfrac{\cos[k(h-z)]}{\sin(kh)}\Big|_{z=h} u(t) = \eta k \dfrac{1}{\sin(kh)}\, u(t) = \eta k \operatorname{cosec}(kh)\, u(t)$

$v_x(z,t)$ oscillates with peaks and troughs at fixed positions; $v_x(h,t) = \phi \ \forall t$

$\rightarrow v_x(z,t) = \alpha e^{i(kz-\omega t)} + \beta e^{i(-kz-\omega t)} = \left[\alpha e^{ikz} + \beta e^{-ikz}\right] e^{-i\omega t}$

with the b.c. on top $\quad v_x(h,t) = \left[\alpha e^{ikh} + \beta e^{-ikh}\right] e^{-i\omega t} = \phi$

$\Rightarrow \alpha e^{ikh} + \beta e^{-ikh} = \phi \ \rightarrow \ \alpha = -\beta e^{-2ikh}$

so let us define $\beta \doteq \frac{\gamma}{2} e^{ikh}$ so $\alpha = -\frac{\gamma}{2} e^{-ikh}$

and the solution can be written as $\quad v_x(z,t) = \frac{\gamma}{2}\left[-e^{ik(z-h)} + e^{-ik(z-h)}\right] e^{-i\omega t} =$

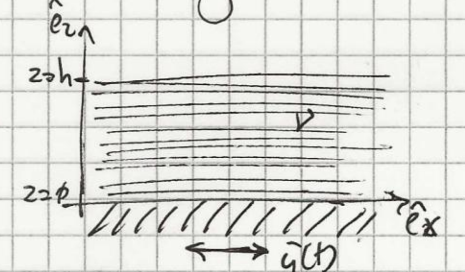
$$= \frac{\gamma}{2}\left[e^{ik(h-z)} - e^{-ik(h-z)}\right] e^{-i\omega t} = i\gamma \sin[k(h-z)]\, e^{-i\omega t}$$

and finally applying the b.c. in $z=\phi$

$v_x(\phi, t) = i\gamma \sin(kh) e^{-i\omega t} = u_0 e^{-i\omega t} \quad \Rightarrow i\gamma = u_0$ and $\boxed{v_x(z,t) = \dfrac{\sin[k(h-z)]}{\sin(kh)}\, u_0 e^{-i\omega t}}$

that is the solution found using the alternative form.

Oscillating viscous fluid layer with a free surface



Here we consider again an incompressible viscous flow with similar features to the previous ones (unbounded in x and y, laying on an oscillating bottom) but unconstrained on top, as a very light fluid (e.g., the atmosphere) comparable to vacuum will exert no appreciable friction on the surface at $z=h$. If the bottom oscillates with periodic motion and velocity $\vec{u}(t) = u_0 e^{-i\omega t}\hat{e}_x$, the usual guess solution $\vec{v}(z,t) = v_x(z,t)\hat{e}_x$ is subject to the system of eqs.

$$\begin{cases} \dfrac{\partial v_x}{\partial t} = \nu \dfrac{\partial^2 v_x}{\partial z^2} & x\text{-component of the Navier-Stokes eq.} \\ v_x(0,t) = u_0 e^{-i\omega t} & \text{no-slip b.c. at the bottom} \\ \sigma_{zx}(h,t) = \eta \left.\dfrac{\partial v_x}{\partial z}\right|_{z=h} = 0 & \text{no-friction b.c. at the free surface} \end{cases}$$

and we try again the form $\vec{v}(z,t) = [A\sin(Kz) + B\cos(Kz)]e^{-i\omega t}\hat{e}_x$,

so that by plugging it into the Navier-Stokes eq. we get the already known dispersion relation

$K^2 = i\omega/\nu$ where we choose $K = +\sqrt{i\omega/\nu}$ i.e. $K = (i+1)/\delta$ $(\delta \doteq \sqrt{2\nu/\omega})$.

The only difference comes out by enforcing the b.c.;

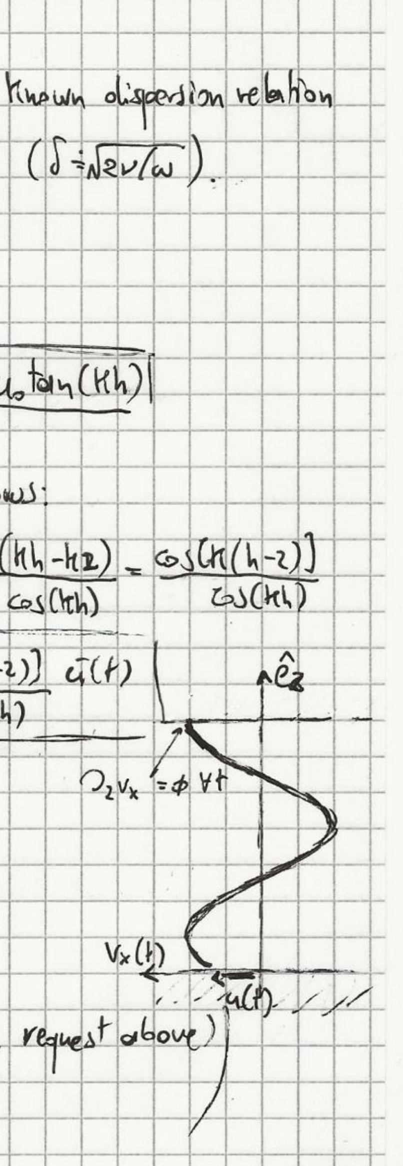
$v_x(0,t) = Be^{-i\omega t} = u_0 e^{-i\omega t} \rightarrow \boxed{B = u_0}$ as before while

$\eta \left.\dfrac{\partial v_x}{\partial z}\right|_{z=h} = \eta K[A\cos(Kh) - u_0\sin(Kh)]e^{-i\omega t} = 0 \rightsquigarrow \boxed{A = u_0 \tan(Kh)}$

The space-dependent part of the solution can be manipulated as follows:

$$\tan(Kh)\sin(Kz) + \cos(Kz) = \frac{\sin(Kh)\sin(Kz) + \cos(Kh)\cos(Kz)}{\cos(Kh)} = \frac{\cos(Kh-Kz)}{\cos(Kh)} = \frac{\cos[K(h-z)]}{\cos(Kh)}$$

so the full solution is $\boxed{\vec{v}(z,t) = \dfrac{\cos[K(h-z)]}{\cos(Kh)} u_0 e^{-i\omega t}\hat{e}_x = \dfrac{\cos[K(h-z)]}{\cos(Kh)}\vec{u}(t)}$



and the friction at the wall (per unit area) is

$$\sigma_{zx}\Big|_{z=0} = \eta \left.\frac{\partial v_x}{\partial z}\right|_{z=0} = \eta K \left.\frac{\sin[K(h-z)]}{\cos(Kh)}\right|_{z=0} u(t) = \eta K \tan(Kh) u(t)$$

$$\left(-\sigma_{zx}\Big|_{z=h} = -\eta \left.\frac{\partial v_x}{\partial z}\right|_{z=h} = -\eta K \left.\frac{\sin[K(h-z)]}{\cos(Kh)}\right|_{z=h} u(t) = 0 \text{ as per b.c. request above} \right)$$

Oscillations of a body of arbitrary shape in a viscous fluid

When the body (solid obstacle to the flow) has no specific symmetry, unlike the previous examples with oscillating plates there is no straightforward way to say $(\bar{v}\cdot\text{grad})\bar{v}=\phi$; let us see what kind of considerations we can make when the advective term can be neglected somehow, and discuss how it can be so. Dropping the advective term, the Navier-Stokes eq. becomes

$$\frac{\partial\bar{v}}{\partial t}=-\frac{1}{\rho}\text{grad}p+\nu\nabla^2\bar{v}$$ and taking the curl of the eq.,

$$\frac{\partial}{\partial t}(\text{curl}\,\bar{v})=\nu\nabla^2(\text{curl}\,\bar{v})$$

We learnt from previous examples (as well as from basic knowledge of mathematics) that the solution to any equation in the form $\partial_t f=\nu\nabla^2 f$ has a solution with f decreasing exponentially with the distance, i.e. in our case

$$\bar{f}=\text{curl}\,\bar{v}\sim e^{-r/\delta}$$

whose meaning is that the motion is rotational around the body, but over a spatial scale δ it becomes essentially irrotational (potential flow: $\text{curl}\,\bar{v}=\phi \Rightarrow \exists\varphi/\bar{v}=\text{grad}\varphi$ which, combined with the incompressibility condition $\text{div}(\bar{v})=\phi$, yields $\nabla^2\varphi=\phi$).

We shall analyze now two cases where the potential flow condition holds, arising from opposite limits in the comparison between δ penetration depth and ℓ characteristic length of the body.

① $\delta\gg\ell$ + $Re\ll1$

If $Re\ll1$, and we call a the body's oscillation amplitude (much smaller that the body size ℓ), ω the oscillation angular frequency, then the body has a velocity $v_0\sim a\omega$ and we assemble a Reynolds number $Re=\frac{a\omega\ell}{\nu}\ll1$

corresponding to a slow-oscillation case (small ω), that is small rate of change for $\bar{v}$; therefore $\partial\bar{v}/\partial t$ can be neglected in the Navier-Stokes equation, while the advective term is ignored as a consequence of $Re\ll1$. All in all, we end up having a practically steady-state flow in the Stokes approximation, i.e. the flow at any time instant is the solution of a <u>time-independent Stokes equation</u> where the body is considered <u>with its instantaneous velocity</u> (and a corresponding drag force proportional to the instantaneous velocity).

(c) $\delta \ll \ell$ + $a \ll \ell$

To ignore the advective term we also need to request $a \ll \ell$ — and thus velocity varies over a length scale that is large with respect to the oscillation amplitude; an order-of-magnitude estimate yields indeed

* in the vicinity of the body, $\bar{v}$ ~ tangential velocity with variation over a scale $\sim \ell$

$\Rightarrow (\bar{v}\cdot \text{grad})\bar{v} \sim v^2/\ell \sim a^2\omega^2/\ell$ (with $v \sim a\omega$)

* $\partial\bar{v}/\partial t \sim \omega v \sim a\omega^2$

$\Rightarrow$ to neglect the advective term we ask for

$$\partial\bar{v}/\partial t \gg (\bar{v}\cdot\text{grad})\bar{v} \quad \text{i.e. } a\omega^2 \gg a^2\omega^2/\ell \quad \text{that is } \underline{\text{indeed } a \ll \ell}$$

while no prescription is necessary concerning Re. Vorticity (curl $\bar{v}$) decays across a thin layer δ around the body and beyond this region we observe an ideal potential flow.

Following Landau's careful considerations, we must notice that we still have a problem with the b.c.: The no-slip condition cannot be satisfied by an ideal fluid (only the normal velocity relative to the body is required to vanish). The solution to the problem under consideration, obtained if we used the ideal potential flow description, would not be acceptable in this thin boundary layer around the body as the tangential velocity would not correspond to the tangential velocity component of the body. In other words, the tangential velocity experiences an abrupt gradient across the boundary layer, connecting the potential flow region to the immersed body.

In order to get a decent description of such a situation, in general terms, we can think of considering a local problem in a region in the proximity of the body over a surface sufficiently larger than δ but also smaller than ℓ, so that we can reduce it to a flat surface and use the oscillating-plate geometry as a ground for the solution, with $\hat{e}_x$ along the surface and $\hat{e}_z$ normal to it (as the surface scale is $> \delta$ we are including in the local domain the correct region of interest while still accepting a plane-geometry approximation). Therefore if we say the velocity found in the potential flow region is $v_0 e^{-i\omega t}$, v_x (the tangential fluid velocity with respect to the body) will decay approaching the body as

$$v_x = v_0 e^{-i\omega t}\left(1 - e^{-(1-i)z/\delta}\right)$$

and the energy dissipation will take the form

$$\langle \dot{E}_{\eta} \rangle = -\frac{1}{2}\sqrt{\omega\eta\rho/2}\oint_{S_b} |v_0^2|\, da \quad \text{with } S_b \text{ surface of the body}$$

(power per unit area in the flat-plate case $-\frac{1}{2}\sqrt{\omega\eta\rho/2}\; u_0^2$ integrated over S_b).

A final remark about the friction force: The expressions we have found in the previous examples are generally complex, and we interpret this as a dissipation (real part) with a phase shift with respect to the forcing (imaginary part). Let us see this in more detail with a general consideration.

As the drag is proportional to the forcing $u = u_0 e^{-i\omega t}$ (a complex expression itself), we can write it, calling <u>β drag coefficient</u> this proportionality factor, as

$$F = \beta u = (\beta_1 + i\beta_2)u \quad \text{with } \beta \in \mathbb{C} \Rightarrow \beta = \beta_1 + i\beta_2 \text{ with } \beta_1, \beta_2 \in \mathbb{R}$$

$$\Rightarrow F = \beta_1 u + i\beta_2 u = \quad \text{using the fact that } \dot{u}(t) = -i\omega u(t)$$

$$= \beta_1 u - \beta_2 \dot{u}/\omega$$

i.e. an expression with real coefficient β_1 and $-\beta_2/\omega$ where F is shown to be the sum of two terms, proportional to a velocity and an acceleration, respectively.

The time-averaged power dissipation is the product of the real parts of drag force and velocity, averaged over an oscillation period.

$$\mathrm{Re}(F) = \frac{1}{2}(F + F^*) = \frac{1}{2}\left(\beta u + (\beta u)^*\right) = \frac{1}{2}\left(\beta u_0 e^{-i\omega t} + \beta^* u_0^* e^{i\omega t}\right)$$

$$\mathrm{Re}(u) = \frac{1}{2}(u + u^*) = \frac{1}{2}\left(u_0 e^{-i\omega t} + u_0^* e^{i\omega t}\right)$$

$$\Rightarrow \langle \dot{E}_{\eta} \rangle = \langle F u \rangle = \left\langle \frac{1}{4}\left[\underbrace{u_0^2 \beta e^{-2i\omega t}} + u_0 u_0^* \beta + u_0 u_0^* \beta^* + \underbrace{u_0^{*2} \beta e^{2i\omega t}}\right] \right\rangle \Rightarrow$$

$\hookrightarrow$ oscillating terms with zero average

$$\underline{\underline{\langle \dot{E}_{\eta} \rangle}} = \frac{1}{4}|u_0|^2(\beta + \beta^*) = \underline{\underline{\frac{1}{2}|u_0|^2 \beta_1}}$$

where we can see that dissipation really does depend only only on the real part of the drag coefficient β and thus only on the velocity, not on the acceleration. The acceleration-induced part of the drag force expression is called <u>inertial part</u> (while the other one is the <u>dissipative part</u>), equivalent to "increase the mass" of the moving (or dragged) object, so to say, and implies a phase shift in the periodic motion (or, if we switch off the perturbation, we can see that the dissipative part tries to dampen the oscillation, while the inertial part is responsible for <u>slowing down the oscillation</u>; this phenomenon is used in viscometers where the decay of an oscillatory motion in a viscous fluid is used to infer the fluid's viscosity. An example of such instrument is the so-called oscillating-disc viscometer, where a viscous fluid occupies the region between two fixed plates, and a disc capable of torsional oscillations is placed in the volume, parallel to the plates).