Elastic waves

We want to discuss here the propagation of waves within elastic media. As a necessarily brief introduction to the subject, the emphasis is on the basic assumptions and features leading to how we set up wave equations in matter.

The starting hypotheses are as follows.

- Transformations are reversible.
- Transformations are adiabatic (hence, the two assumptions say they are isentropic).
- We may consider as infinitesimal
  - displacements
  - density perturbations with respect to equilibrium, and their time/space derivatives

so that as a consequence not only $\bar{v}(\bar{x},t) = \partial_t \bar{u}(\bar{x},t)$ is exact, but also

* $\bar{a}(\bar{x},t) = \partial^2 u(\bar{x},t)/\partial t^2$ (negligible advective acceleration)

* $\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho_e \bar{v})$ continuity eq.

* $\rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = \operatorname{div} \bar{\bar{\sigma}}$ or equivalently $\rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = (\mu+\lambda)\operatorname{grad}\operatorname{div}\bar{u} + \mu \nabla^2 \bar{u}$

Navier eq. (eq. of motion) without volume forces ($\bar{g}$ and Coriolis), which we shall prove to be negligible.

Negligible advective acceleration

Let us see that we can ignore the advective (inertial) term of the acceleration if the displacement velocity of the continuum element is negligible with respect to the wave propagation velocity and the displacement amplitude is negligible with respect to the dimensions of the medium. We shall call

$A \rightarrow$ oscillation amplitude scale

$v \rightarrow$ continuum element's typical velocity

$a_E \rightarrow$ explicit acceleration's $(\partial \bar{v}/\partial t)$ scale

$a_I \rightarrow$ advective acceleration's $((\bar{v}\cdot\operatorname{grad})\bar{v})$ scale

$T \rightarrow$ time scale of variations in amplitude and velocity (for a wave, a good measure is the wave period)

$\ell \rightarrow$ length scale over which amplitude and velocity vary significantly.

So we can say $v \sim A/T = A\omega \Rightarrow a_E \sim v/T \sim A/T^2$,

$$\operatorname{grad}\bar{v} \sim v/l \sim A/lT \Rightarrow a_I = \bar{v}\cdot\operatorname{grad}\bar{v} \sim v^2/l \sim A^2/lT^2$$

and $\bar{a}_I$ is negligible if $a_I \ll a_E$

$$\Rightarrow \quad a_I/a_E \sim \frac{v^2}{l}\cdot\frac{T}{v} \sim \frac{vT}{l} \ll 1 \iff \underline{v \ll l/T}$$

$$\text{or} \quad a_I/a_E \sim \frac{A^2}{lT^2}\cdot\frac{T^2}{A} \sim \frac{A}{l} \ll 1 \iff \underline{A \ll l}$$

Let us interpret these inequalities. $c \doteq l/T$ has the dimensions of a velocity, and it represents a characteristic velocity scale in the medium, that is not the displacement velocity $\partial\bar{u}/\partial t$; if we consider waves, the length scale l to consider is $l = \min(\lambda, L)$ the smallest between wavelength λ and linear dimension L of the continuum; T, as we said already, is reasonably a wave period. Hence:

- if $\lambda < L \Rightarrow l = \lambda$, the condition $v \ll l/T = \lambda/T = c$ means "$v \ll c$ where for waves we shall see that c = wave propagation speed";
- if $L < \lambda \Rightarrow l = L$: the condition $A \ll l = L$ means small oscillation amplitude.

Either condition must be met to make inertial acceleration negligible. Acoustic and seismic waves lie in the condition $\lambda \ll L$ and element displacement velocity $v \ll c$ wave speed, which complies with the requirement for negligible a_I.

Negligible volume forces for waves with not-too-small frequency

The complete equation for elastodynamics reads

$$\rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = \operatorname{div}\bar{\bar{\Sigma}}_1 + \bar{F}_{v_1}, \quad \text{i.e.} \quad \rho_e \frac{\partial^2 \bar{u}}{\partial t^2} = (\mu+\lambda)\operatorname{grad}\operatorname{div}\bar{u} + \mu\nabla^2\bar{u} + \rho'\bar{g} + 2\rho_e \bar{v}\times\bar{\Omega}$$

and we may consider the last two terms (body forces) to be negligible for waves of frequency that is not "too small". Let us note first that the lower limit is set by seismic waves, whose lower frequency bound is around 0.1 Hz (below this we can find essentially some low-frequency "seismic hum", while dominant contributions are in the 0.1–10 Hz range). Let us evaluate some orders of magnitude first.

$$\frac{\mu+\lambda}{\rho_e}\operatorname{grad}\operatorname{div}\bar{u}, \ \frac{\mu}{\rho_e}\nabla^2\bar{u} \sim c^2 U/\lambda^2 = \nu^2 U \quad \text{magnitude of surface forces per unit mass}$$

where we have estimated the coefficients $(\mu+\lambda)/\rho_e \sim \mu/\rho$ to be $\sim c^2$ wave speed squared (a fact we shall prove better later), and U, ν wave amplitude and frequency, respectively.

Concerning the density perturbation ρ', since $\partial_t \rho' = -\mathrm{div}(\rho_e \bar{v})$, $\Rightarrow$

$$\rho' \omega \sim \rho_e U\omega/\lambda \Rightarrow \rho'/\rho_e \sim U/\lambda .$$

⊙ The Coriolis force is estimated as $2\bar{v} \times \bar{\Omega} \sim U\omega\Omega$,
with Ω angular rotation frequency of the Earth, such that $F = \Omega/2\pi = 1/(24 \cdot 3600) \simeq 10^{-5}\,Hz$.

Hence

$$\frac{\text{Coriolis force}}{\text{surface force}} \sim \frac{U\omega\Omega}{\nu^2 U} = \frac{(2\pi)^2 F}{\nu} \simeq 4.6 \cdot 10^{-4}/\nu \ll 1 \quad \text{for } \nu \geq 0.1\,Hz$$

that is, the Coriolis term is absolutely negligible.

Note: In phenomena where the Coriolis force cannot be neglected, since its form is of the type $\bar{v} \times \bar{\Omega}$ with a vector product that exchanges cross-components, this term will mix longitudinal and transverse displacement components, therefore playing a big role in the physics of the concerned process (e.g., transverse displacements enter the game of otherwise purely longitudinal matters and vice versa).

⊙ Gravity force per unit mass is estimated as

$$\rho'/\rho_e \bar{g} \sim \rho'/\rho_e\, g \sim Ug/\lambda ,$$

hence

$$\frac{\text{gravity force}}{\text{surface force}} \sim \frac{Ug}{\lambda}\frac{1}{\nu^2 U} = g/\lambda\nu^2 \underset{\lambda\nu = c}{=} \frac{g}{c\nu} \simeq \frac{10}{10^3 \cdot 0.1} \sim 10^{-1} \text{ still} < 1;$$

even in a conservative limit of slow ($c = 10^3\,m/s$), low-frequency ($\nu = 0.1\,Hz$) seismic waves, gravity has a modest, when not negligible, relevance (the assumption may be a bit crude). Once again, when $\bar{g}$ cannot be neglected, it has a mixing effect of components, a longitudinal-transverse coupling since in the gravity perturbation $\rho'/\rho_e \bar{g}$ the density perturbation ρ' appears, governed by the continuity equation where all the components of the velocity of the continuum element occur; as a consequence, they will come up in all components of the wave.

Formal deduction of wave equations for elastic waves

Let us see how we can get wave-like equations from the Navier equation, i.e. describing wave motion of elastic media. We present two cases.

⊙ P waves

Let us take the divergence of the Navier eq.; since the order of time and space derivatives can be swapped, we get

$$\rho \frac{\partial^2}{\partial t^2} \operatorname{div} \bar{u} = (\mu+\lambda) \underbrace{\operatorname{div}[\operatorname{grad}}_{\nabla^2}(\operatorname{div}\bar{u})] + \mu \nabla^2 \operatorname{div} \bar{u}$$

and calling $\vartheta \doteq \operatorname{div}\bar{u}$

$$\rho \frac{\partial^2 \vartheta}{\partial t^2} = (\mu+\lambda) \nabla^2 \vartheta + \mu \nabla^2 \vartheta \Rightarrow \boxed{\frac{\partial^2 \vartheta}{\partial t^2} = \left(\frac{2\mu+\lambda}{\rho}\right) \nabla^2 \vartheta} \quad (*)$$

we obtain a three-dimensional d'Alembert equation, i.e. a wave equation, for the variable $\vartheta = \operatorname{div}\bar{u}$. If we recall that the divergence of $\bar{u}$ is associated to an isotropic expansion/contraction, it becomes natural to state that this must be the <u>general eq. of pressure waves</u> (or P waves) in an elastic solid, describing a wave behaviour of the <u>volumetric deformation (or strain)</u> $\vartheta = \operatorname{div}\bar{u}$.

⊙ S waves

If we apply the curl to the Navier eq. instead, we get

$$\rho \frac{\partial^2}{\partial t^2} \operatorname{curl} \bar{u} = (\mu+\lambda) \underbrace{\operatorname{curl}[\operatorname{grad}}_{\operatorname{curl}(\operatorname{grad}) = \emptyset}(\operatorname{div}\bar{u})] + \mu \nabla^2 \operatorname{curl} \bar{u}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial t^2} \operatorname{curl} \bar{u} = \frac{\mu}{\rho} \nabla^2 \operatorname{curl} \bar{u}} \quad (**)$$

again a wave eq., but for the quantity $\operatorname{curl}\bar{u}$, which is associated to rotation and shear and indeed called <u>shear deformation (or strain)</u>; hence this is the <u>general eq. of shear waves (or S waves)</u> in an elastic solid.

Notice that for the eqs. (*), (**) to represent waves, the coefficients on the right-hand side must be associated with the wave velocity:

$$\frac{2\mu+\lambda}{\rho}, \quad \frac{\mu}{\rho} \sim c^2$$ (a fact we used in the estimates for the proof of negligible volume forces).

In the following we see two simple, one-dimensional examples of P and S waves.

Pressure waves (1D).

Let us consider an elastic bar of cylindrical cross section and infinite length along the $\hat{e}_x$ axis. We assume infinitesimal displacement along $\hat{e}_x$, uniform across the transverse cross section at any position x; hence $\underline{\vec{u} = u_x(x,t)\hat{e}_x}$.

The deformation causes internal stresses that result in the insurgence of waves for which we derive an equation. The components of the strain tensor $\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$, since only $u_x(x,t)$ is nonzero yield

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \partial_x u & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{pmatrix} \quad \text{with} \quad Tr(\underline{\underline{\varepsilon}}) = \varepsilon_{nn} = \partial_x u(x,t);$$

hence $\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}} + \lambda\varepsilon_{nn}\underline{\underline{\mathbb{1}}}$ reads

$$\underline{\underline{\sigma}} = \begin{pmatrix} (2\mu+\lambda)\partial_x u & \phi & \phi \\ \phi & \lambda\partial_x u & \phi \\ \phi & \phi & \lambda\partial_x u \end{pmatrix}$$

where we can see, besides a stress along the longitudinal deformation, that normal stresses appear in transverse directions. Now let us write the Navier eq. for u_x; since $\mathrm{div}\,\vec{u} = \partial_x u_x$,

$$\rho\frac{\partial^2 u_x}{\partial t^2} = (\mu+\lambda)\partial_x^2 u_x + \mu\nabla^2 u_x \quad \Rightarrow \quad \text{since} \quad \nabla^2 u_x = \partial_x^2 u_x,$$

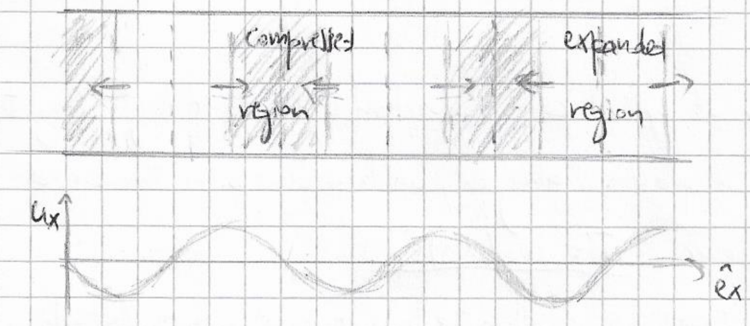
$$\rho\frac{\partial^2 u_x}{\partial t^2} = (2\mu+\lambda)\frac{\partial^2 u_x}{\partial x^2}$$

while for u_y $\quad \rho\frac{\partial^2 u_y}{\partial t^2} = (2\mu+\lambda)\partial_y\partial_x u_y + \mu\nabla^2 u_y = \phi \quad$ and similarly for u_z;

the x-component u_x is the only one that is not identically zero and thus we have a one-dimensional D'Alembert eq. $\boxed{\frac{\partial^2 u_x}{\partial t^2} = \frac{2\mu+\lambda}{\rho}\frac{\partial^2 u_x}{\partial x^2}}$ with $v_p = [(2\mu+\lambda)/\rho]^{1/2}$ wave velocity

which is a 1D PRESSURE (P) WAVE because it describes periodic expansions and contractions along the longitudinal direction; it is a longitudinal wave, travelling along the direction of the oscillation (oscillating displacement) itself, that is parallel to the inducing force.

In terms of the phenomenological elastic coefficients, $\frac{\partial^2 u_x}{\partial t^2} = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}\frac{\partial^2 u_x}{\partial x^2}$.



Sketch of 1D P wave (contraction/expansion is exaggerated: infinitesimal variations of $\bar{u}$, ρ)

Shear waves (1D)

We consider the same bar but here we have only displacements orthogonal to the $\hat{e}_x$ axis, let us say along the $\hat{e}_y$ transverse direction, i.e. $\bar{u} = u_y(x,t)\hat{e}_y = u(x,t)\hat{e}_y$. Then

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \Rightarrow \underline{\underline{\varepsilon}} = \begin{pmatrix} \phi & \frac{1}{2}\partial_x u & \phi \\ \frac{1}{2}\partial_x u & \phi & \phi \\ \phi & \phi & \phi \end{pmatrix} \quad ; \quad Tr(\underline{\underline{\varepsilon}}) = \phi$$

(isochoric, volume-preserving deformation)

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}} + \lambda \underbrace{Tr(\underline{\underline{\varepsilon}})}_{\phi}\underline{\underline{1}} = 2\mu\underline{\underline{\varepsilon}} = \mu\begin{pmatrix} \phi & \partial_x u & \phi \\ \partial_x u & \phi & \phi \\ \phi & \phi & \phi \end{pmatrix}$$

i.e. basically $\underline{\underline{\sigma}}$ reproduces $\underline{\underline{\varepsilon}}$, direct stress-strain correspondence.

We might recall here that $\mu = G$ shear modulus; also, one could prove that μ is the same for isothermal and adiabatic transformations.

As this is a volume-preserving strain ($\text{div}\,\bar{u} = \varepsilon_{kk} = \phi$), the Navier eq. becomes

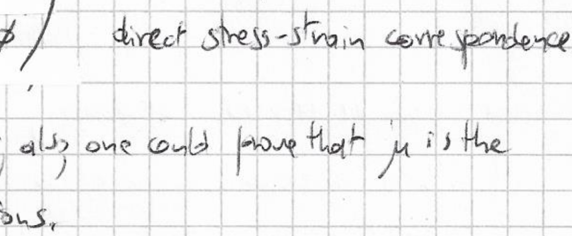
$$\rho\frac{\partial^2 u_i(x,t)}{\partial t} = \mu\nabla^2 u_i \quad \text{and explicitly}$$

$$i = x \Rightarrow \rho\frac{\partial^2}{\partial t^2}u_x = \mu\nabla^2 u_x = \phi; \quad i = z \Rightarrow \rho\frac{\partial^2 u_z}{\partial t^2} = \mu\nabla^2 u_z = \phi;$$

$$i = y \Rightarrow \rho\frac{\partial^2 u_y}{\partial t^2} = \mu\nabla^2 u_y = \mu\frac{\partial^2 u_y}{\partial x^2} \Rightarrow \boxed{\frac{\partial^2 u_y}{\partial t^2} = \frac{\mu}{\rho}\frac{\partial^2 u_y}{\partial x^2}}$$

with $\frac{\mu}{\rho} = \frac{E}{\rho 2(1+\nu)} = v_s^2$ wave velocity

eq. for a 1D SHEAR (S) WAVE, called like this as continuum elements slide perpendicularly to the wave propagation (transverse wave).



Sketch of 1D S wave

Some important notes:

- As wave eqs. are partial differential eqs., the solution requires boundary conditions (e.g., for an infinite bar we must have finite displacements for $x \to \pm\infty$) and initial conditions;

and since the d'Alembert eq. is of the second order in t, we need to know both $f(x,\phi)$ and $\partial_t f(x,\phi)$ (with $f(x,t)$ the solution $\forall x, t$).

⊙ We have used linear approximations that make our wave equations linear; since the boundary condition are typically (almost always) linear as well, the superposition principle holds and thus a linear combination of solutions is also a solution.

⊙ Elastic waves propagate momentum and energy, but <u>not</u> matter (continuum elements oscillate around equilibrium positions).

⊙ As with any other transverse waves, S waves can be <u>polarized</u> in different ways (polarization indicates the axis of oscillation).

Negligible advective acceleration – a reprise

The "basic block" of the solution for the d'Alembert eq. has the form

$$u(x,t) = A\cos(kx-\omega t) \quad \text{(yielding a trivial dispersion relation } k=\omega/c)$$

so we can reprise the conditions to neglect the advective acceleration; indeed

$$v(x,t) = \partial_t u(x,t) = A\omega \sin(kx-\omega t) \sim A\omega \quad \text{in magnitude}$$

$$\partial_t v(x,t) = -A\omega^2 \cos(kx-\omega t) \quad \sim A\omega^2$$

$$\partial_x v(x,t) = A\omega k \cos(kx-\omega t) = \frac{2\pi A\omega}{\lambda}\cos(kx-\omega t) \sim 2\pi A\omega/\lambda .$$

Hence when we say $\bar{a}(x,t) = \partial_t \bar{v}(x,t) + (\bar{v}\cdot \text{grad})\bar{v}$

and we compare the order of magnitude of the two (explicit and inertial) terms, and want to neglect the second one we are asking for

$$A\omega^2 \gg A\omega \cdot 2\pi A\omega/\lambda$$

i.e. $2\pi A/\lambda \ll 1$, or $\boxed{A \ll \lambda}$

and we recover the condition of small oscillation amplitude (with respect to λ).

Adiabatic vs isothermal conditions for elastic waves

For frequencies that are not too high, elastic waves are shown to be adiabatic.

We define a thermalization time $\tau(\lambda)$ for a monochromatic wave of wavelength λ, which is the time required to damp (due to conduction) the temperature perturbation in a piece of the continuum of length λ due to the wave. It represents a diffusion time, hence it has a form

$\tau \simeq L^2/\chi$ with χ thermal diffusivity, so for our case

$$\tau(\lambda) \simeq \lambda^2/\chi \simeq c^2T^2/\chi \quad \text{as } \lambda = cT \text{ (T period, c wave velocity).}$$

We also define a characteristic time of the medium

$$\tau_c \doteq \chi/c^2 \quad \Rightarrow \tau(\lambda) \simeq c^2T^2/c^2\tau_c \simeq T^2/\tau_c .$$

If $T \ll \tau(\lambda)$, heat transport cannot play a significant role and the process is therefore adiabatic (no heat transfer). If we also assume reversibility, the process is isentropic.

Now take care: The chain of reasoning tells us

$$T \ll \tau(\lambda) \sim T^2/\tau_c \Rightarrow 1 \ll T/\tau_c \Rightarrow \boxed{T \gg \tau_c}$$

which is the opposite of what we seemed to be saying above (i.e, adiabatic process if T is small): The wave period must be long to have an adiabatic process.

On the contrary, if $T \gg \tau(\lambda) \sim T^2/\tau_c \Rightarrow 1 \gg T/\tau_c \Rightarrow \boxed{T \ll \tau_c}$

is the condition where heat conduction distributes (damps) temperature perturbations ($T \ll \tau(\lambda)$ thermalization time): The wave period must be short for an isothermal process.

In terms of frequencies

$T \gg \tau_c \quad \sim \quad f_w \ll f_c \quad$ adiabatic elastic wave;

$T \ll \tau_c \quad \sim \quad f_w \gg f_c \quad$ isothermal elastic wave.

Note: The velocity c is related to the μ, λ coefficients, so we should choose whether they are the isothermal or adiabatic ones; here only an order of magnitude is required for the discussion, hence the choice is not very dramatic; isothermal values are preferred.

Let us make an estimate. For a very slow wave propagation, $c = 10^3$ m/s; on the contrary, let us have high χ to set a limit case, $\chi = 2 \cdot 10^{-4}$ m²/s (as a reference, $\chi \sim 1.7 \cdot 10^{-6}$ for the very good thermal conductor silver). Hence we have a very lower-limit case

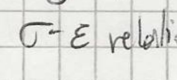
$$f_c \simeq \frac{c^2}{\chi} \simeq 5 \cdot 10^9 \text{ Hz}$$ which is a very high frequency anyway!

Hence in reality we have material with characteristic frequencies higher than this by one or two orders of magnitude and we conclude that elastic waves, and especially seismic waves, are never isothermal but almost perfectly adiabatic (there are approximations, so we say "almost").

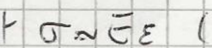
Beyond pure (linear) elasticity - Viscoelastic models

Going beyond the region of elastic behaviour, a more complex response is found where the $\sigma-\varepsilon$ relationship owes both to solid and fluid features. For elastic solids we have that $\sigma \sim E\varepsilon$ (proportionality between stress and strain), while for a viscous Newtonian fluid $\sigma \sim \eta\dot{\varepsilon}$ (stress proportional to strain rate).

The two features are analogous to the behaviour of a spring and a damper, respectively:

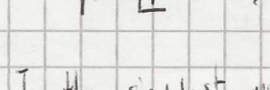


spring ~ elastic $\sigma \sim E\varepsilon$ $(\Rightarrow \dot{\sigma} \sim E\dot{\varepsilon})$

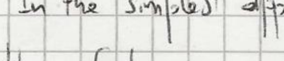


damper (dashpot) ~ viscous $\sigma \sim \eta\dot{\varepsilon}$

In the simplest approximation, we can build up a couple models distinguished by the way these features are combined into a VISCOELASTIC BEHAVIOUR: The Kelvin-Voigt and the Maxwell material models.



Kelvin-Voigt model $\sigma = E\varepsilon + \eta\dot{\varepsilon}$ (*)



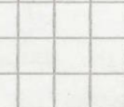
Maxwell model $\dot{\varepsilon} = \dot{\sigma}/E + \sigma/\eta$ (**)

Another useful analogy is the mechanical - electrical one where spring $\leftrightarrow$ resistor R, damper $\leftrightarrow$ capacitor C; it follows intuitively that $\sigma \leftrightarrow$ current I, $\varepsilon \leftrightarrow$ voltage V and the relations (*), (**) are easily seen with the laws for combining electrical impedance in parallel (Kelvin-Voigt) and series (Maxwell):

KV $\rightarrow$ total current = sum of the individual ones on the two branches

$$I = I_R + I_c = \frac{1}{R}V + C\dot{V}$$
$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Maxwell $\rightarrow$ total voltage drop = sum of the two voltages in series

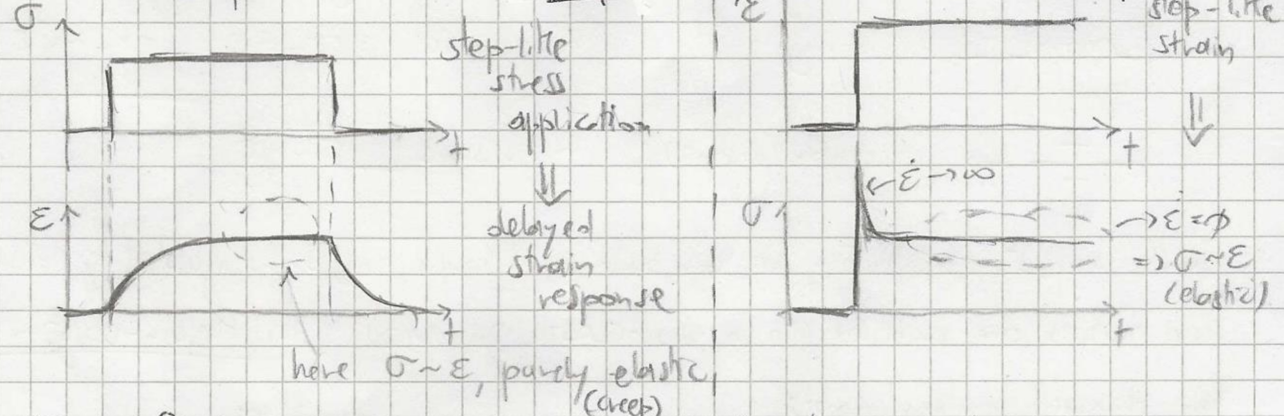


$$V = V_R + V_C = RI + \frac{1}{C}\int I dt \qquad (I = I_R = I_c)$$
$$\Rightarrow \dot{V} = \dot{V}_R + \dot{V}_c = R\dot{I} + \frac{1}{C}I$$
$$\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} + \frac{1}{\eta}\sigma$$

We can get a feeling of the behaviour in the two cases studying the response to simple forms of applied stress or strain, using the electric circuit analogy.

Kelvin-Voigt model $\sigma = E\varepsilon + \eta\dot{\varepsilon}$

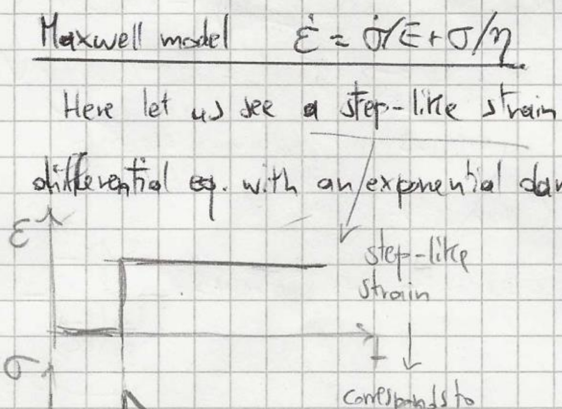
For instance, switching on impulsively a constant stress (like a current step) causes a delayed strain (voltage) response ($V(t) \sim 1-e^{-t/\tau}$ with $\tau = RC \to \eta/E$ the time constant depending on the viscosity/elasticity ratio) until a purely elastic response is reached within a time period of few τ (**creep**: tendency to slow deformation under persistent stress).



If we flip the reasoning and imagine a step-like strain, at that moment $\dot{\varepsilon} \to \infty$ so in the relation $\sigma = E\varepsilon + \eta\dot{\varepsilon}$ we get a spike in σ, then $\dot{\varepsilon} = \emptyset$ and $\sigma = E\varepsilon$ (elastic behaviour again). In conclusion, a KV material behaves more like an elastic solid, apart from a transient response characterized by viscosity.

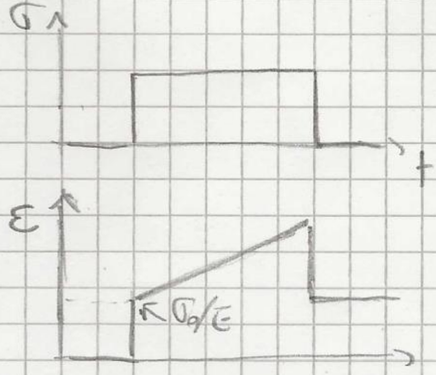
Maxwell model $\dot{\varepsilon} = \dot{\sigma}/E + \sigma/\eta$

Here let us see a step-like strain; after the step, $\dot{\varepsilon} = \emptyset \Rightarrow \dot{\sigma}/E + \sigma/\eta = \emptyset$, simple differential eq. with an exponential damping as solution, that is, the stress decays on a relaxation time $\tau = \eta/E$: $\sigma(t) = E\varepsilon e^{-t/\tau}$.



If we had a constant strain rate $\dot{\varepsilon}$ the stress would increase and saturate: $\sigma(t) \sim \eta\dot{\varepsilon}(1-e^{-t/\tau})$ ($t \to \infty \Rightarrow \sigma \sim \eta\dot{\varepsilon}$: like a fluid).

Looking the other way round, i.e., applying a step-like stress from zero to a constant value σ_0, since $\dot{\sigma} = \emptyset$ after the transient $\dot{\varepsilon} = \sigma/\eta$ constant strain rate and by integration $\varepsilon(t) \sim \sigma_0/E + \sigma_0/\eta \, t$; if we release the stress, the elastic part springs back to the initial state, but the viscous (fluid-like) part does not:



We conclude that this is more like a fluid, there is no creep like in the KV material, but an irreversible deformation (stopping the stress does not interrupt the deformation).