Hydrostatics of incompressible fluids

For incompressible fluids, $\rho =$ constant, the hydrostatic condition

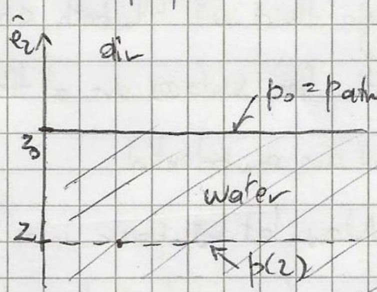
$$-\frac{1}{\rho}\,\mathrm{grad}\,p + \vec{g} = \phi \qquad \text{with } \vec{g} = -g\hat{e}_z$$

is straightforwardly integrated: $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \phi$; $\frac{\partial p}{\partial z} = -\rho g \Rightarrow$

$$\boxed{p(z) = -\rho g z + \text{constant}} \quad \text{or} \quad \boxed{p(z) + \rho g z = \text{constant}} \qquad \text{STEVIN'S LAW}$$

or again equivalently if we have a reference height z_0 where $p = p_0$ is known (e.g. the free surface of a fluid basin, where $p_0 = p_{atm}$),

$$p(z) - p(z_0) = \rho g (z_0 - z) = \rho g h \qquad \text{with } h \doteq z_0 - z$$

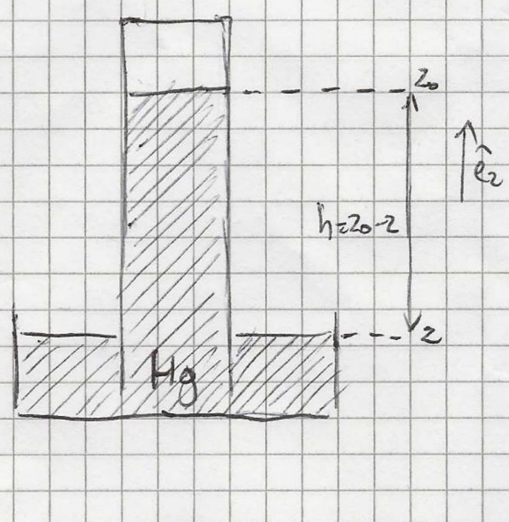


Example: Going underwater the pressure increases (see figure →); we have an increase of 1 atm $\simeq 10^5$ Pa when $10^5 = \rho_{H_2O} \cdot g \cdot h \simeq 10^3 \cdot 10 \cdot h \Rightarrow$ for $h = 10$ m; the pressure increases by 1 atm every 10 m of depth, with critical implications on scuba diving as the human physiology is severely affected.

Let us see some more examples.

Torricelli's barometer

It is the first instrument that can measure atmospheric pressure. A long tube with one of its ends closed is filled with mercury (a high density fluid) and turned upside down into a wider tank without letting Hg to spill. More Hg is poured into the tank and the pipe is lifted a bit. Some Hg will flow out, leaving an empty volume on top of the pipe (vacuum); hence the Hg level in the pipe (height z_0) will be at pressure $p(z_0) = \phi$ (no force pushing down Hg any more). Applying Stevin between z_0 and z = level of Hg in the tank,



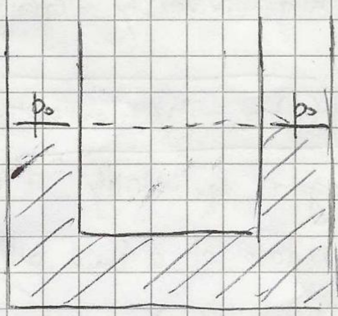
$$\underbrace{p(z)}_{p_{atm}} - \underbrace{p(z_0)}_{\phi} = \rho g (z_0 - z) = \rho g h$$

$$\Rightarrow p_{atm} = p(z) = \rho g h$$

By experimental observation, $h = 0.76$ m ; since $\rho_{Hg} = 13.6 \cdot 10^3$ kg/m^3, $\Rightarrow \underline{p_{atm} \simeq 1.013 \cdot 10^5 \text{ Pa}}$.

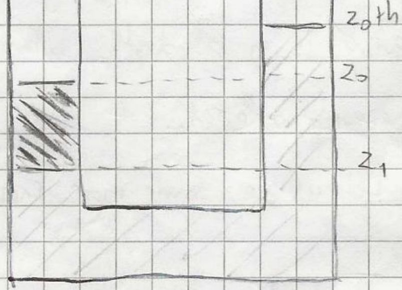
[The pressure unit Torr comes from here ; It represents the pressure exerted by a column of height 1 mm of Hg, therefore it is 1/760 atm, or equivalently 1 atm = 760 Torr (torr: name of the unit; Torr: symbol).]

Communicating vessels - immiscible fluids



Let us consider a U-shaped pipe as in the figure, → i.e. having the two branches = communicating vessels. A single fluid will fill both arms at the same level, since both free surfaces are at the same (atmospheric) pressure p_0 and are equipotential.

Now let us pour into the left arm an immiscible fluid with different density $\rho_2 \neq \rho_1$ (we shall do it carefully so that no chaotic mixing occurs). The two levels will change. Let us impose Stevin's law for the two arms (see new figure →)



RIGHT: $p^{(1)}(z_1) - p_{atm} = \rho_1 g (z_0 + h - z_1)$

LEFT : $p^{(2)}(z_1) - p_{atm} = \rho_2 g (z_0 - z_1)$ *

Continuity of pressure requires $p^{(1)}(z_1) = p^{(2)}(z_1)$

$\Rightarrow \quad \rho_1 (z_0 + h - z_1) = \rho_2 (z_0 - z_1)$

$\Rightarrow \quad h = \frac{\rho_2}{\rho_1}(z_0 - z_1) - (z_0 - z_1) \Rightarrow h = \left(\frac{\rho_2}{\rho_1} - 1\right)(z_0 - z_1)$

↗ $h > \emptyset$ if $\rho_2 > \rho_1$ (as in the figure)

↘ $h < \emptyset$ if $\rho_2 < \rho_1$

*Note: We are neglecting here the term $\rho_{air} g h$ and saying that since the air density ρ_{air} is $\ll \rho_1, \rho_2$ we have the same p_{atm} at both free surfaces, even if they are at a different height. The proper equality should be

$$\rho_1 g (z_0 + h - z_1) = \rho_2 g (z_0 - z_1) + \rho_{air} g h \Rightarrow$$

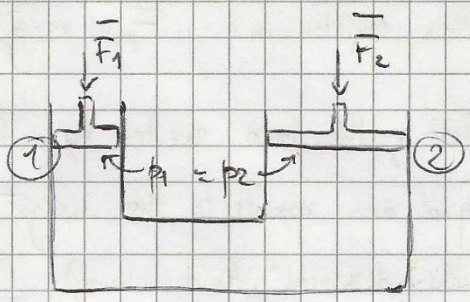
$$h + z_0 - z_1 = \frac{\rho_2}{\rho_1}(z_0 - z_1) + \frac{\rho_{air}}{\rho_1} h \Rightarrow$$

$$h (1 - \rho_{air}/\rho_1) = \left(\frac{\rho_2}{\rho_1} - 1\right)(z_0 - z_1) \Rightarrow h = \frac{1}{1 - \rho_{air}/\rho_1}\left(\frac{\rho_2}{\rho_1} - 1\right)(z_0 - z_1)$$

(very close to the approximate result for $\rho_{air}/\rho_1 \to \emptyset$).

Hydraulic press

Now let us have two communicating pipes with very different cross-sectional areas $A_1 < A_2$. When filled with a fluid, $p_1 = p_2$ in the two pipes at equilibrium; if we apply the forces $\bar{F}_1, \bar{F}_2$ on the two surfaces



$$p_1 = \frac{F_1}{A_1} = \frac{F_2}{A_2} = p_2 \Rightarrow F_1 = \frac{A_1}{A_2} F_2$$

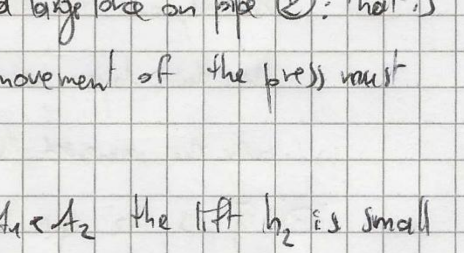
that is, a small force on the pipe 1 can balance a large force on pipe 2. That is the working principle of hydraulic lifts. Obviously a movement of the press must displace the same fluid volume in the two pipes,

$$\Delta V_1 = \Delta V_2 \Rightarrow h_1 A_1 = h_2 A_2$$ and for $A_1 < A_2$ the lift h_2 is small (i.e. we can lift with a small force but it takes a while).

Archimedes' principle

What is the force exerted by a fluid at rest on the part of a body that is immersed in the fluid? [Careful: The result is valid only in the hydrostatic case while forces will differ in a dynamic situation.]

Each surface element of the immersed part is under the effect of the fluid's pressure, i.e. an element of area da at position $\bar{x}$ with outward-pointing normal $\hat{n}$ experiences $-p(\bar{x})\hat{n}\,da$; the resulting total force will be



$$\bar{F}_A = -\int_S p(\bar{x})\hat{n}\,da \quad \text{with } S \text{ wetted surface}$$

whose explicit calculation would require the knowledge of the exact body geometry. Yet we can argue that $\bar{F}_A$ is the same force that would be exerted on S wetted surface if instead of V_i, volume of the immersed part of the body, we replaced it with the same volume of fluid (still at rest). This is what is called the "displaced" fluid in Archimedes' principle

"The upward buoyant force exerted on a body, totally or partially immersed in a fluid, is equal to the weight of the fluid displaced by the body."

Explicitly, the resultant of Archimede's force $\overline{F}_A$ and gravity $\overline{F}_G$ must be zero to attain equilibrium: $\overline{F}_A + m_f \overline{g} = \phi \Rightarrow \underline{\overline{F}_A = -m_f \overline{g}}$ where $m_f = \rho_f V_i$

that is, the force exerted by a fluid at rest on the immersed portion of the object is equal and opposite to the weight of the fluid that would otherwise occupy that volume (the "displaced" fluid volume).

Notes: ⊙ As a consequence the force depends only on the immersed part of the body. This holds also if the container where we place fluid and object is too shallow, and the fluid volume is less than what could be "displaced" if the fluid layer were deeper; that is because $\overline{F}_A$ is the integral of pressure over the immersed (wetted) surface.

⊙ $\overline{g}$ is the gravitational force observed in the reference frame; in an accelerated (non-inertial) reference, fictitious forces (e.g., centrifugal forces in a rotating system) must be taken into account.

Center of buoyancy and equilibrium

Equilibrium requires not only the resultant of forces to be zero but also the total torque $\overline{T}$ (moment of force) to be zero. Let us consider the torque with respect to a generic point O (pivot).

$\overline{F}_G$ is applied to the center of mass G $\Rightarrow \overline{T}_{w,b} = (\overline{x}_G - \overline{x}_0) \times m_b \overline{g}$

weight, body

The pressure on the immersed part of the body yields a pressure torque $\overline{T}_p$.

Now let us look at the "displaced fluid"; $\overline{F}_G$ on it yields a torque

$\overline{T}_{w,f} = (\overline{x}_s - \overline{x}_0) \times m_f \overline{g}$ where $\overline{x}_s$ = center of mass of the displaced fluid; again, the pressure on the ideal surface separating the displaced fluid from the fluid surrounding it yields a pressure torque $\overline{T}_p$ equal to the one above.

Let us require zero torque for the fluid to ensure static equilibrium:

$$\overline{T}_p + \overline{T}_{w,f} = \phi \Rightarrow \overline{T}_p = -(\overline{x}_s - \overline{x}_0) \times m_f \overline{g} \underset{\text{by Archimede's principle}}{=} (\overline{x}_s - \overline{x}_0) \times \overline{F}_A$$

which means the torque of pressure forces on the immersed body (with respect to any pivot point O) is equal to the torque of Archimede's buoyancy force applied in the

CENTER OF BUOYANCY, i.e. the center of mass of the displaced fluid.

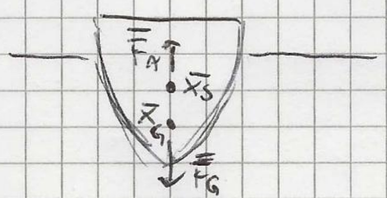
Let us reason again about the body and its equilibrium, which requires

$$\bar{T}_p + \bar{T}_{w,b} = \phi \quad \text{i.e.}$$

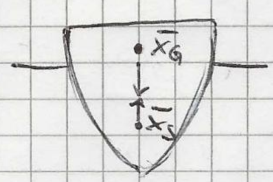
$$(\bar{x}_G - \bar{x}_0) \times m_b \bar{g} - (\bar{x}_S - \bar{x}_0) \times m_f \bar{g} = \phi$$

i.e. necessarily
$$(\bar{x}_G - \bar{x}_S) \times \bar{g} = \phi$$

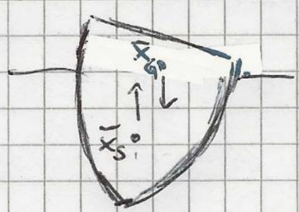
that is, the center of mass of the body and the center of buoyancy must lie along the direction of gravity (i.e., along a vertical line in an inertial reference frame). The equilibrium will be stable if $\bar{x}_G$ is below $\bar{x}_S$ (see figure ↓).



stable equilibrium



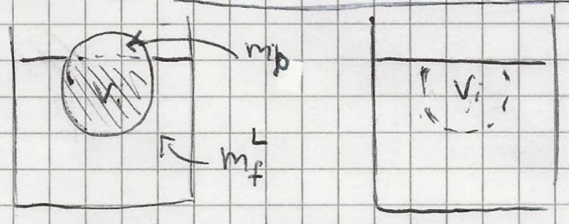
unstable equilibrium



out of equilibrium

Isostasy

Consider two equal containers, with the same fluid inside. The fluid level is the same, but one of the container also contains a floating object. In this condition (floating object, same fluid level) the masses in the two containers are equal.



On the left, the total mass is $m^L = m_b + m_f^L$

i.e. $\quad m^L = m_b(-\rho_f V_i + \rho_f V_i) + m_f^L$

On the right, $\quad m^R = m_f^L + \rho_f V_i$

$$\Rightarrow \quad m^L = m_b - \rho_f V_i + m^R$$

but since the body is floating, the mass of the "displaced fluid" $\rho_f V_i$ gives rise to a buoyancy force equal to gravity on the body, i.e. $m_b = \rho_f V_i$

$$\Rightarrow \quad \underline{m^L = m^R}.$$