Cartesian tensors [Theorem # as per Segel's Chapter 2]

Definition: An n-th order (or n-th rank) <u>tensor</u> $\underline{\underline{T}}$, with $n = \cancel{0}, 1, 2, \dots$, is an object such that

- In any Cartesian cs there is a rule for associating $\underline{\underline{T}}$ to a unique and ordered set of 3^n quantities $T_{i_1, i_2, i_3, \dots, i_n}$ called <u>components</u> of $\underline{\underline{T}}$;
- If $T_{i_1, \dots, i_n}$ and $T'_{j_1, \dots, j_n}$ are the components of $\underline{\underline{T}}$ in two different cartesian <u>coordinate systems</u> => there is a transformation law
$$\boxed{T_{i_1, \dots, i_n} = l_{i_1 j_1} \dots l_{i_n j_n} T'_{j_1, \dots, j_n}}$$ with $\underline{\underline{l}}$ orthogonal transformation.

Example: 2nd order tensor $\rightarrow$ $T_{pq} = l_{pm} l_{qn} T'_{mn}$

3rd order tensor $\rightarrow$ $T_{pqr} = l_{pm} l_{qn} l_{rk} T'_{mnk}$

4th order tensor $\rightarrow$ $T_{pqrs} = l_{pm} l_{qn} l_{rk} l_{sh} T'_{mnkh}$

...

Convention: $(\underline{\underline{T}})_{ij} = T_{ij}$ is the ij-th component of $\underline{\underline{T}}$ in a given cs

Notable tensors:
- δ_{ij} 2nd order tensor
- ε_{ijk} 3rd order tensor (to be precise, pseudotensor or tensor density)
- Some tensors are special on the grounds of them being <u>ISOTROPIC</u>.

An <u>isotropic tensor</u> is a tensor whose components stay the same for any rotation of the cs.

Zero-order tensors, i.e. SCALARS, are all isotropic.

First-order tensors, i.e. VECTORS: no isotropic 1st order tensors exist (beside the <u>null</u> vector).

Second-order tensors: The only isotropic one is δ_{ij} (or its multiples $\alpha\delta_{ij}$, with $\alpha \in \mathbb{R}$).

We can easily prove it is isotropic: $\delta'_{ij} = l_{mi} l_{nj} \delta_{mn} = l_{mi} l_{mj} = l^{-1}_{im} l_{mj} = \delta_{ij}$

(see Appendix 1 for a proof of its uniqueness)

Third-order tensors: the only one being isotropic is ε_{ijk} (and its multiples $\alpha\varepsilon_{ijk}$, with $\alpha \in \mathbb{R}$).

The space of tensors of any order n has a vector space structure, i.e. it is endowed with typical operations that must be valid for any coordinate system. Some obvious operations and properties already well known for scalars and vectors. Let us see them for order 2 tensors:

$(\alpha \underline{\underline{T}})_{ij} = \alpha T_{ij}$ (multiplication by a scalar number)

$(\underline{\underline{T}} + \underline{\underline{U}})_{ij} = T_{ij} + U_{ij}$ (sum) and clearly $(\underline{\underline{T}} - \underline{\underline{U}})_{ij} = T_{ij} - U_{ij}$ (subtraction)

$\underline{\underline{T}} + (-\underline{\underline{T}}) = \underline{\underline{\emptyset}}$ (for each tensor, there exists its opposite)

and the null tensor $\underline{\underline{\emptyset}}$ exists, i.e. a tensor for which $[\underline{\underline{\emptyset}}]_{ij} = \emptyset$ (all components are null in any CS)

All these properties show that this is a linear space

(we could synthetically express them at once as $(\alpha \underline{\underline{T}} + \beta \underline{\underline{U}})_{ij} = \alpha T_{ij} + \beta U_{ij}$).

There is more than this: other operations are defined for tensors.

A CONTRACTION of an n-th order tensor, with $n \geq 2$, is defined as the operation obtained by setting two indices of the tensor equal and then performing the sum on all of their range of values (as dummy indices). The result of the contraction is a tensor of order n-2. Let us see it, for instance, when $n=4$ $\rightarrow$

Theorem 4: Let $\underline{\underline{T}}$ be a fourth-order tensor with components T_{ijmn}; then setting $m=j$ and defining $U_{in} = T_{ijjn}$ $\Rightarrow$ U_{in} is a second-order tensor.

Proof: If $\underline{\underline{T}}$ is a tensor of order 4, its transformation rule is

$$T_{ijmn} = \ell_{ip}\ell_{jq}\ell_{mr}\ell_{ns} T'_{pqrs} \quad ; \quad \text{setting} \quad m=j$$

$$T_{ijjn} = \ell_{ip}\underbrace{\ell_{jq}\ell_{jr}}_{= \ell^{-1}_{qj}\ell_{jr} = \delta_{qr}}\ell_{ns} T'_{pqrs} = \ell_{ip}\ell_{ns}\delta_{qr} T'_{pqrs} = \ell_{ip}\ell_{ns} T'_{prrs}$$

So $\quad U_{in} = T_{ijjn} = \ell_{ip}\ell_{ns} T'_{prrs} = \ell_{ip}\ell_{ns} U'_{ps}$

and this is indeed the transformation law for a tensor of order 2. q.e.d.

Note: The transformation law for 2nd-order tensors (a transformation under an $O(3)$ orthogonal group), can also be written in another familiar form i.e. $\underline{\underline{T}}' = \underline{\underline{\ell}}\,\underline{\underline{T}}\,\underline{\underline{\ell}}^{-1}$; indeed

$$T_{ij} = \ell_{ih}\ell_{jk} T'_{hk} = \ell_{ih} T'_{hk} \ell^{T}_{kj} = \ell_{ih} T'_{hk} \ell^{-1}_{kj} = (\underline{\underline{\ell}}\,\underline{\underline{T}}'\,\underline{\underline{\ell}}^{-1})_{ij}, \text{ that is } \underline{\underline{T}} = \underline{\underline{\ell}}\,\underline{\underline{T}}'\,\underline{\underline{\ell}}^{-1}$$

The TENSOR PRODUCT (or OUTER product) is defined as follows for $\underline{\underline{A}}$ n-th order tensor and $\underline{\underline{B}}$ m-th order tensor, i.e. tensors with components

$$[\underline{\underline{A}}]_{i_1, i_2, \dots, i_n} = A_{i_1, i_2, \dots, i_n}$$
$$[\underline{\underline{B}}]_{i_1, i_2, \dots, i_m} = B_{i_1, i_2, \dots, i_m}$$

the tensor product $\underline{\underline{A}} \otimes \underline{\underline{B}}$ has components $A_{i_1, \dots, i_n} B_{j_1, \dots, j_m}$, and thus it is a tensor of order $m+n$ (as stated by Theorem 5).

As an example, given two vectors (1st-order tensors) $\bar{v}, \bar{w}$ we have a tensor product $\bar{v} \otimes \bar{w}$ yielding a 2nd-order tensor with components

$(\bar{v} \otimes \bar{w})_{ij} = v_i w_j$, or in full

$$(\bar{v} \otimes \bar{w}) = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix}$$

Similarly the tensor product of $\bar{v}$ vector and $\underline{\underline{T}}$ 2nd-order tensor is $(\bar{v} \otimes \underline{\underline{T}})_{ijk} = v_i T_{jk}$ (order 3) etc.

Note: Saying $\bar{v} \otimes \bar{w}$ is a tensor implies that $(\bar{v} \otimes \bar{w})_{ij} = v_i w_j$ is an expression valid for any cs, which can be verified:

$$(\bar{v} \otimes \bar{w})_{ij} = l_{ih} l_{jk} (\bar{v} \otimes \bar{w})'_{hk} = l_{ih} l_{jk} v'_h v'_k = l_{ih} v'_h \, l_{jk} v'_k = v_i w_j$$

Note: the tensor product of $\bar{v}, \bar{w}$ yields a 2nd-order tensor T_{ij}; yet it is not true that any 2nd-order tensor can be obtained from the tensor product of two vectors.

What can be done instead is taking the canonical basis for vectors, $\hat{e}_1 = (1, \phi, \phi)$, $\hat{e}_2 = (\phi, 1, \phi)$, $\hat{e}_3 (\phi, \phi, 1)$ and using it to build a basis that can be exploited to write $\underline{\underline{T}}$ as a linear combination of the basis elements.

An element of such basis is $\hat{e}_i \otimes \hat{e}_j$ (therefore the basis is made out of 9 2nd-order tensors)

e.g. $\hat{e}_1 \otimes \hat{e}_2 = \begin{pmatrix} \phi & 1 & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{pmatrix} \Rightarrow \underline{\underline{T}} = \sum_{i,j=1}^{3} c_{ij} (\hat{e}_i \otimes \hat{e}_j)$ ($c_{ij} = T_{ij}$ in usual notation)

As much as we can contract two indices of a tensor, we can define an operation called <u>CONTRACTION PRODUCT</u> (or INNER product) between two tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$; for instance, if these are 2nd-order tensors, their inner product, indicated by the symbol $\cdot$, is

$[\underline{\underline{A}} \cdot \underline{\underline{B}}]_{iq} = A_{ij} B_{jq}$ which is indeed a contraction of $\underline{\underline{A}} \otimes \underline{\underline{B}}$ over two adjacent indices.

The contraction product can be performed on tensors of any order, always by making equal (and summing over) two adjacent indices. Given two tensors of order m and n, respectively, their inner product has order $m+n-2$ ($\rightarrow$ Theorem 6).(*)

Another important property of tensors is the <u>quotient rule</u> (presented in Segel's Theorem 7 and further): If $\underline{v}$ and $\underline{w}$ are tensors of order n and m, and an object $\underline{\underline{T}}$ such that $\underline{\underline{v}} = \underline{\underline{T}} \cdot \underline{\underline{w}}$ and this expression holds $\forall$ cs (invariant under coordinate transformation), then $\underline{\underline{T}}$ is a tensor of order $n+m$.

(*) = For 2nd-order tensors (matrices), the contraction product $\underline{\underline{A}} \cdot \underline{\underline{B}}$ is the usual product of matrices A and B.

In particular, if $\bar{v}$ and $\bar{w}$ are vectors and the expression $v_i = T_{ij} w_j$ holds $\forall$ cs $\Rightarrow$ T_{ij} is a second-order tensor. Let us prove this case.

The expression between $\bar{v}, \bar{w}, \underline{\underline{T}}$ is written for two cs as

$$v_i' = T_{ij}' w_j' \quad ; \quad v_i = T_{ij} w_j \; ;$$

the transformation law for vectors states $v_i' = \ell_{ki} v_k$, $w_j' = \ell_{hj} w_h$

$\Rightarrow$ in $v_i' = T_{ij}' w_j'$ we replace v_i', w_j' and get

$$\ell_{ki} v_k = T_{ij}' \ell_{hj} w_h \quad ; \text{ by multiplication by } \ell_{ip}^{-1} \text{ of both sides,}$$

$$\underbrace{\ell_{ki} \ell_{ip}^{-1}} v_k = T_{ij}' \ell_{ip}^{-1} \ell_{hj} w_h$$

$$\delta_{kp} v_k = v_p = T_{ij}' \ell_{pi} \ell_{hj} w_h \quad \text{and we can also write } v_p = T_{ph} w_h$$

$$\Rightarrow T_{ph} w_h = T_{ij}' \ell_{pi} \ell_{hj} w_h \quad ; \text{ since } w_h \text{ is chosen arbitrarily,}$$

$\underline{T_{ph} = T_{ij}' \ell_{pi} \ell_{hj}}$ holds which is the transformation law for 2nd-order tensors

q.e.d.

Finally, a <u>SCALAR PRODUCT</u> is also defined in a tensor space, and it is bilinear and commutative (i.e., it is a true scalar product). The scalar product between tensors $\underline{\underline{T}}, \underline{\underline{U}}$ with components $T_{i_1 i_2 \dots i_n}$ and $U_{i_1 i_2 \dots i_n}$ ($\underline{\underline{T}}$ and $\underline{\underline{U}}$ must be of the same order) is

$$T_{i_1 \dots i_n} U_{i_1 \dots i_n} \quad \text{(all indices become dummy)}$$

Ex.: for 1st-order tensors = vectors $\bar{t}, \bar{u}$ the scalar product is $t_i u_i$ as already well known; for 2nd-order tensors T_{ij}, U_{ij} the scalar product reads $T_{ij} U_{ij}$.

Decomposition of second-order tensors

A 2nd-order tensor $\underline{\underline{S}}$ is said to be SYMMETRIC if $\underline{\underline{S}}^T = \underline{\underline{S}}$ (i.e. $S_{ij} = S_{ji}$).

A 2nd-order tensor $\underline{\underline{A}}$ is said to be ANTISYMMETRIC (or SKEW-SYMMETRIC) if $\underline{\underline{A}}^T = -\underline{\underline{A}}$ (i.e. $A_{ij} = -A_{ji}$).

A symmetric tensor has only 6 different components at most, so the space of symmetric tensors has dimension = 6. (or subspace in the space of 2nd-order tensors)

An antisymmetric tensor has only 3 different components at most (the out-of-diagonal ones are 6 and the 3 upper ones are opposite to the 3 lower ones), so the space of antisymmetric tensors has dimension 3.

The scalar product between a symmetric tensor and an antisymmetric one is null: $S_{ij} A_{ij} = \phi$ (easily seen due to how their components are defined); hence we can say that the two subspaces of symmetric and antisymmetric tensors are ORTHOGONAL and their sum yields the whole space of 2nd-order tensors. These subspaces are <u>proper</u> subspaces, i.e. they are neither the whole space nor the zero subspace.

A fundamental property of the space of 2nd-order tensors is the fact that any such tensor $\underline{\underline{T}}$ can be decomposed into the sum of a symmetric tensor and and an antisymmetric one,

$$\underline{\underline{T}} = \underline{\underline{S}} + \underline{\underline{A}},$$

and this decomposition is <u>unique</u>. This decomposition is easily shown:

$$T_{ij} = \frac{1}{2} T_{ij} + \frac{1}{2} T_{ij} + \frac{1}{2} T_{ji} - \frac{1}{2} T_{ji} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) = S_{ij} + A_{ij}$$

where $S_{ij} \doteq \frac{1}{2}(T_{ij} + T_{ji})$ is symmetric (swapping i and j we get the same value, so the transpose matrix is equal to the original one: $S^T_{ij} = S_{ji} = S_{ij}$)

and $A_{ij} \doteq \frac{1}{2}(T_{ij} - T_{ji})$ is antisymmetric (swapping i and j we get $\frac{1}{2}(T_{ji} - T_{ij}) = -\frac{1}{2}(T_{ij} - T_{ji})$, so transpose is opposite of original matrix: $A^T_{ij} = A_{ji} = -A_{ij}$)

The subspace of 2nd-order symmetric tensors itself is made out of the sum of two proper subspaces, which are cs-invariant and orthogonal one to each other:

- $\underline{\underline{S}}^{TL}$ traceless ($Tr(\underline{\underline{S}}^{TL}) \equiv S^{TL}_{ii} = \phi$) symmetric tensors (subspace of dimension 5)
- $\underline{\underline{I}}$ isotropic symmetric tensors, proportional to δ_{ij} (identity matrix): $\underline{\underline{I}} = \alpha \underline{\underline{1}} = \alpha \delta_{ij}$ (dimension 1)

Let us also see this further decomposition: Subtract from $\underline{\underline{S}}$ a matrix equal to the identity times $\frac{1}{3}$ of the trace, i.e. $\frac{1}{3} S_{\ell\ell}\delta_{ij} = \frac{1}{3} Tr(\underline{\underline{S}})\underline{\underline{\mathbb{1}}}$. Note that the trace of this matrix is

$$\frac{1}{3} \cdot Tr(\underline{\underline{S}})\, Tr(\underline{\underline{\mathbb{1}}}) = \frac{1}{3} Tr(\underline{\underline{S}}) \cdot 3 = Tr(\underline{\underline{S}}) \Rightarrow$$

so that:

$$\underline{\underline{S}} = \underbrace{\underline{\underline{S}} - \frac{1}{3} Tr(\underline{\underline{S}})\underline{\underline{\mathbb{1}}}} + \underbrace{\frac{1}{3} Tr(\underline{\underline{S}})\underline{\underline{\mathbb{1}}}}$$

or

$$S_{ij} = \underbrace{S_{ij} - \frac{1}{3} S_{\ell\ell}\delta_{ij}} + \underbrace{\frac{1}{3} S_{\ell\ell}\delta_{ij}} = S^{TL}_{ij} + I_{ij}$$

All in all a 2nd-order tensor $\underline{\underline{T}}$ can be decomposed (in a unique way) into 3 parts:

$$\underline{\underline{T}} = \underline{\underline{S}}^{TL} + \underline{\underline{A}} + \underline{\underline{I}}$$

traceless symmetric — antisymmetric — symmetric and isotropic

Let us explicitly decompose the symmetric part of T_{ij}, i.e. $\frac{1}{2}(T_{ij} + T_{ji})$:

$$\frac{1}{2}(T_{ij} + T_{ji}) = \frac{1}{2}\left[T_{ij} + T_{ji} - \frac{1}{3}(T_{\ell\ell} + T_{\ell\ell})\delta_{ij} + \frac{1}{3}(T_{\ell\ell} + T_{\ell\ell})\delta_{ij}\right] =$$

$$= \frac{1}{2}\left[T_{ij} + T_{ji} - \frac{2}{3} T_{\ell\ell}\delta_{ij}\right] + \frac{1}{3} T_{\ell\ell}\delta_{ij}$$

$$\Rightarrow \boxed{T_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3} T_{\ell\ell}\delta_{ij}\right) + \frac{1}{2}(T_{ij} - T_{ji}) + \frac{1}{3} T_{\ell\ell}\delta_{ij}}$$

$$T_{ij} = \underbrace{\qquad}_{S^{TL}_{ij}} + \underbrace{\qquad}_{A_{ij}} + \underbrace{\qquad}_{I_{ij}}$$

It is worth recalling explicitly some facts already mentioned or hinted at above:

- The three components are cs-invariant, i.e. each transforms independently upon cs rotation (a traceless symmetric tensor transforms into another traceless symmetric one, and so on; if a component is zero in a cs, it will be so $\forall$ cs).
- One can prove the decomposition is unique.
- Linear combinations of tensors of one kind yields tensors of the same kind (each subspace is linear, so linear combination is an operation <u>within</u> the subspace).
- The trace of a 2nd-order tensor is an invariant quantity. Indeed, given the transformation law $T_{ij} = \ell_{im}\ell_{jn}T'_{mn}$,

$$\Rightarrow \quad T_{ii} = \ell_{im}\ell_{in}T'_{mn} = \ell^{-1}_{mi}\ell_{in}T'_{mn} = \delta_{mn}T'_{mn} = T'_{mm}$$

The eigenvalue problem and principal axes

Let us focus on 2nd-order tensors. We could ask ourselves, when is $\underline{\underline{T}}$ diagonal, i.e. such that $T_{ij} = \phi \ \forall i \neq j$? The question is physically relevant when we apply it to the stress tensor, that is a tensor describing the forces acting on a continuum element. In this context, the question means: Is there a direction (CS rotation) such that stresses are purely normal, i.e. orthogonal to the surface element around a certain point (so that the action is purely 'press or pull', without shear i.e. force along the surface)?

We can write formally such condition as follows: Take this tensor $\underline{\underline{T}}$ and a vector $\bar{v}$ in the desired direction; hence the condition is

$$\bar{v} \cdot \underline{\underline{T}} = \lambda \bar{v} \qquad \text{(the result is a vector } \lambda\bar{v} \text{ along } \bar{v} \text{ itself)} \quad (*)$$

and this is quite apparently an eigenvalue problem: In components,

$$v_i T_{ij} = \lambda v_j \rightarrow v_i T_{ij} = \lambda v_i \delta_{ij} \rightarrow v_i \left(T_{ij} - \lambda \delta_{ij} \right) = \phi$$

i.e. a set of 3 equations for the 3 unknown quantities v_i, the eigenvalue equations. $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ (scalars) are the eigenvalues $\lambda^{(p)}$ (not necessarily $\lambda^{(1)} \neq \lambda^{(2)} \neq \lambda^{(3)}$) associated to the 3 linearly independent eigenvectors $\bar{v}^{(p)}$. Excluding the trivial solution $v_i = \phi \ \forall i$, a solution exists if $\det(T_{ij} - \lambda \delta_{ij}) = \phi$; so

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = \phi \quad \text{i.e.} \quad -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = \phi \quad \text{characteristic equation}$$

where $I_{1,2,3}$ are called principal invariants

$$I_1 = T_{ii} = \mathrm{Tr}(T_{ij}) \ ; \ I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} \ ; \ I_3 = \det(T_{ij})$$

With corresponding eigenvalue $\lambda^{(p)}$ and eigenvector $\bar{v}^{(p)}$, the required condition (*) reads

$$v_i^{(p)} T_{ij} = \lambda^{(p)} v_j^{(p)} \quad \text{or} \quad v_i^{(p)} T_{ij} = \lambda^{(p)} v_i^{(p)} \delta_{ij} \quad \text{(where there is no sum on } p\text{)}$$

Notice that condition (*) could also be written as $\underline{\underline{T}} \cdot \bar{w} = \lambda \bar{w}$; yet the equations for the eigenvectors' components, when written down explicitly, are not the same and we must speak of left and right eigenvectors. But, interestingly enough, if $\underline{\underline{T}}$ is symmetric then left and right eigenvectors coincide. Furthermore a real symmetric tensor ($T_{ij} \in \mathbb{R} \ \forall i,j$) has real and distinct eigenvalues whose associated eigenvectors are orthogonal. So we can come to the conclusion that for real, symmetric tensors of order 2 $\underline{\underline{T}}$ the following holds:

The cs where $\underline{\underline{I}}$ is diagonal is called <u>principal axis system.</u> In such cs, the orthogonal eigenvectors $\vec{v}^{(i)}$ can be used to build a basis $\hat{e}_i'$ of the cs and

$$\underline{\underline{I}} = \begin{pmatrix} \lambda^{(1)} & \phi & \phi \\ \phi & \lambda^{(2)} & \phi \\ \phi & \phi & \lambda^{(3)} \end{pmatrix}$$

Tensor fields

As much as a scalar (or vector) field is a function that associates a scalar (or vector) to each point in space, a tensor field associates a tensor to each point in space. So this tensor is no longer a (tensor) constant but a function (indeed, a function of position called field) and its properties must be indicated for any position; see, e.g., the transformation law:

$$T_{ij}(P) = \ell_{ih}\ell_{jk} T'_{hk}(P) \qquad \text{i.e.} \qquad T_{ij}(\bar{x}(P)) = \ell_{ih}\ell_{jk} T'_{hk}(\bar{x}'(P))$$

Concepts like limit, continuity, derivatives are defined in the same way they are for scalar/vector fields. So we can define differential operators divergence, gradient, curl for tensors; for 2nd-order tensors $\underline{\underline{T}}$,

$$\text{grad}\,\underline{\underline{T}} : \quad [\text{grad}\,\underline{\underline{T}}]_{pij} = \partial_p T_{ij} \qquad (\text{also written, in 'comma notation', } T_{ij,p} \,;\; {}_{,p} = \partial_p)$$

$$\text{div}\,\underline{\underline{T}} : \quad [\text{div}\,\underline{\underline{T}}]_j = \partial_i T_{ij} \qquad (T_{ij,i})$$

$$\text{curl}\,\underline{\underline{T}} : \quad [\text{curl}\,\underline{\underline{T}}]_{iq} = \varepsilon_{ijp}\partial_j T_{pq} \qquad (\varepsilon_{ijp} T_{pq,j})$$

⊙ If $\underline{\underline{T}}$ is of order m, grad $\underline{\underline{T}}$ is a tensor field of order $m+1$.

Proof for $m=2$: We must prove the transformation law

$$[\text{grad}\,\underline{\underline{T}}]'_{kmn} = \ell_{pk}\ell_{im}\ell_{jn}[\text{grad}\,\underline{\underline{T}}]_{pij}, \quad \text{also written as}$$

$$\frac{\partial}{\partial x'_k} T'_{mn}(\bar{x}') = \ell_{pk}\ell_{im}\ell_{jn}\left[\frac{\partial}{\partial x_p} T_{ij}(\bar{x})\right]_{\bar{x}=\bar{x}(\bar{x}')}$$

$$\text{but } \frac{\partial}{\partial x'_k} T'_{mn}(\bar{x}') = \ell_{im}\ell_{jn}\frac{\partial}{\partial x'_k} T_{ij}(\bar{x}(\bar{x}')) \quad \rightarrow \quad = \ell_{pk}\ell_{im}\ell_{jn}\frac{\partial}{\partial x_p} T_{ij}(\bar{x})$$

$$\text{and } \frac{\partial}{\partial x'_k} T_{ij}(\bar{x}(\bar{x}')) = \left.\frac{\partial}{\partial x_p} T_{ij}(\bar{x})\right|_{\bar{x}=\bar{x}(\bar{x}')} \cdot \frac{\partial x_p}{\partial x'_k} = \ell_{pk}\frac{\partial}{\partial x_p} T_{ij}(\bar{x})$$

$$\left(\text{since } x_p = \ell_{pk} x'_k \Rightarrow \frac{\partial x_p}{\partial x'_k} = \ell_{pk}\right)$$

⊙ If a tensor field $\underline{\underline{T}}$ is of order m, div $\underline{\underline{T}}$ is a tensor field of order $m-1$.

Ex.: For a vector field $\bar{v}$, div($\bar{v}$) is a scalar. Indeed

$$\partial'_h v'_h(\bar{x}') = \partial'_h \ell_{ih} v_i(\bar{x}) = \ell_{ph}\partial_p \ell_{ih} v_i(\bar{x}) = \ell_{ph}\ell^{-1}_{hi}\partial_p v_i(\bar{x}) = \delta_{pi}\partial_p v_i(\bar{x}) = \partial_i v_i(\bar{x})$$

so $\partial'_h v'_h(\bar{x}') = \partial_i v_i(\bar{x})$ that is the transformation law for a scalar (i.e., cs-invariant)

Vector - tensor duality relations

It can be shown that any 2nd-order antisymmetric tensor $\underline{\underline{A}}$ has an associated vector $\bar{\omega}$ (its <u>dual vector</u>) and can be expressed in terms of it as follows:

$$\underline{A_{ij} = \varepsilon_{ijk}\,\omega_k} \qquad \text{with inverse relationship} \qquad \underline{\omega_k = \tfrac{1}{2}\varepsilon_{klm} A_{lm}}$$

Proof: Let us multiply both sides of $\omega_k = \frac{1}{2}\varepsilon_{klm} A_{lm}$ by ε_{ijk}

$$\Rightarrow \quad \varepsilon_{ijk}\,\omega_k = \frac{1}{2}\varepsilon_{ijk}\underbrace{\varepsilon_{klm}}_{\varepsilon_{lmk}} A_{lm} = \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)A_{lm} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = A_{ij}$$

antisymmetric: $A_{ji} = -A_{ij}$

$$\Rightarrow \quad A_{ij} = \varepsilon_{ijk}\,\omega_k \qquad \text{q.e.d.}$$

- The vector $\bar{\omega}$ is the contraction product between tensors $\underline{\underline{A}}$ and $\underline{\underline{\varepsilon}}$ $\left(\bar{\omega} = \frac{1}{2}\underline{\underline{\varepsilon}}\cdot\underline{\underline{A}}\right)$; viceversa, $\underline{\underline{A}}$ is the contraction product of $\underline{\underline{\varepsilon}}$ and $\bar{\omega}$ $\left(\underline{\underline{A}} = \underline{\underline{\varepsilon}}\cdot\bar{\omega}\right)$.
- To be precise, $\bar{\omega}$ is a <u>pseudovector</u>, since it is the contraction product between the pseudotensor $\underline{\underline{\varepsilon}}$ and the tensor $\underline{\underline{A}}$.
- Explicitly, with $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$, then $\underline{\underline{A}} = \begin{pmatrix} \phi & \omega_3 & -\omega_2 \\ -\omega_3 & \phi & \omega_1 \\ \omega_2 & -\omega_1 & \phi \end{pmatrix}$.
- Let us see a physics example. If the velocity vector $\bar{v}$ of the points in a body satisfies the expression $\bar{v} = \bar{x}\cdot\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}\cdot\bar{x}$ for a 2nd-order antisymmetric tensor $\underline{\underline{\Omega}}$ that is independent of of the position $\bar{x}$,
then the motion of this body is a rigid rotation around the cs origin with an angular velocity $\bar{\omega}$ that is the dual pseudovector associated to $\underline{\underline{\Omega}}$, i.e. $\omega_i = \frac{1}{2}\varepsilon_{ijk}\Omega_{jk}$.

Proof: If $\bar{v} = \bar{x}\cdot\underline{\underline{\Omega}}$, in components

$$v_j = x_i\,\Omega_{ij} \underset{\uparrow}{=} x_i\,\varepsilon_{ijk}\,\omega_k = \varepsilon_{kij}\,\omega_k x_i = (\bar{\omega}\times\bar{x})_j$$

using $\underline{\underline{\Omega}}$'s dual $\bar{\omega}$

so $\quad v_j = (\bar{\omega}\times\bar{x})_j \quad$ that is $\quad \bar{v} = \bar{\omega}\times\bar{x} \quad$ that is indeed the expression of a rigid rotation with angular velocity $\bar{\omega}$.

q.e.d.

Appendix 1 - Isotropy of δ_{ij} (proof of uniqueness)

Let $\underline{\underline{T}}$ be a generic 2nd-order tensor of components T_{ij} in a CS $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$.
Also, let us suppose $\underline{\underline{T}}$ is isotropic (i.e. its components stay the same $\forall$ CS), and let's see what happens upon rotation of the CS around $\hat{e}_3$ by an angle $\pi/2$, i.e.

$\hat{e}_1' = \hat{e}_2$; $\hat{e}_2' = -\hat{e}_1$; $\hat{e}_3' = \hat{e}_3$ which is accomplished by a rotation matrix

$$\underline{\underline{\ell}} = \begin{pmatrix} \phi & 1 & \phi \\ -1 & \phi & \phi \\ \phi & \phi & 1 \end{pmatrix}$$ so that the transformation applied to $\underline{\underline{T}}$ is $\underline{\underline{T}}' = \underline{\underline{\ell}}\,\underline{\underline{T}}\,\underline{\underline{\ell}}^{-1}$:

$$\begin{pmatrix} T_{11}' & T_{12}' & T_{13}' \\ T_{21}' & T_{22}' & T_{23}' \\ T_{31}' & T_{32}' & T_{33}' \end{pmatrix} = \begin{pmatrix} \phi & 1 & \phi \\ -1 & \phi & \phi \\ \phi & \phi & 1 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} \phi & -1 & \phi \\ 1 & \phi & \phi \\ \phi & \phi & 1 \end{pmatrix} = \ldots = \begin{pmatrix} T_{22} & -T_{21} & T_{23} \\ -T_{12} & T_{11} & -T_{13} \\ T_{32} & -T_{31} & T_{33} \end{pmatrix}$$

Isotropy requires $T_{ij}' = T_{ij} \quad \forall i,j \Rightarrow T_{11} = T_{11}' = T_{22}$

$T_{31} = T_{31}' = T_{32} = T_{32}' = -T_{31}$; $T_{31} = -T_{31} \Leftrightarrow T_{31} = \phi$

$T_{32} = T_{32}' = -T_{31} = -T_{31}' = -T_{32}$; $T_{32} = -T_{32} \Leftrightarrow T_{32} = \phi$

Similarly, a $\pi/2$ rotation around $\hat{e}_2$ will yield the following equalities (to satisfy isotropy):

$T_{11} = T_{33}$; $T_{12} = T_{32} = \phi$; $T_{21} = T_{23} = \phi \Rightarrow T_{ij} = \phi$ if $i \neq j$

T_{ii} have the same value, say α:

$$T_{ij} = \begin{pmatrix} \alpha & \phi & \phi \\ \phi & \alpha & \phi \\ \phi & \phi & \alpha \end{pmatrix} = \alpha \underline{\underline{\mathbb{1}}}$$ also written as $T_{ij} = \alpha \delta_{ij}$

so δ_{ij} is proven to be the only isotropic 2nd-order tensor.

Appendix 2 – $\det(\underline{\underline{1}} + \phi \underline{\underline{A}}) = 1 + \phi \, Tr(\underline{\underline{A}})$ if ϕ scalar of infinitesimal magnitude

⊛ From Segel's Chapter 1, Theorem 11

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$; let us make $j=q$ and sum over the dummy index,

$$\varepsilon_{ijk}\varepsilon_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}$$

Let us repeat with $i=p$ $\Rightarrow$

(scalar product) $\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{ii} = 6$ (*)

⊛ From Segel's Chapter 1, Theorem 9

$$\varepsilon_{ijk} A_{ir} A_{js} A_{kt} = \varepsilon_{ijk} A_{ri} A_{sj} A_{tk} = \varepsilon_{rst} \det \underline{\underline{A}}$$

Let us multiply both sides of the equality by ε_{rst} and sum over the dummy indices r, s, t:

$$\varepsilon_{rst}\varepsilon_{ijk} A_{ir} A_{js} A_{kt} = \varepsilon_{rst}\varepsilon_{rst} \det \underline{\underline{A}} = 6 \det \underline{\underline{A}} \quad (**)$$

⊛ Let us have $\underline{\underline{B}} \doteq \underline{\underline{1}} + \phi \underline{\underline{A}}$ where ϕ is a scalar of infinitesimal magnitude;

now we know that, by (**),

$$\det(\underline{\underline{B}}) = \det(\underline{\underline{1}} + \phi \underline{\underline{A}}) = \frac{1}{6} \varepsilon_{ijl}\varepsilon_{hkm} B_{ih} B_{jk} B_{lm} =$$

$$= \frac{1}{6} \varepsilon_{ijl}\varepsilon_{hkm} (\delta_{ih} + \phi A_{ih})(\delta_{jk} + \phi A_{jk})(\delta_{lm} + \phi A_{lm}) =$$ to first order in ϕ (infinitesimal)

$$= \frac{1}{6} \varepsilon_{ijl}\varepsilon_{hkm} \left[\delta_{ih}\delta_{jk}\delta_{lm} + \phi \left(\delta_{ih}\delta_{jk} A_{lm} + \delta_{ih} A_{jk} \delta_{lm} + A_{ih} \delta_{jk} \delta_{lm} \right) \right]$$

Now we can say – $\varepsilon_{ijl}\varepsilon_{hkm}\delta_{ih}\delta_{jk}\delta_{lm} = \varepsilon_{ijl}\varepsilon_{ijl} = 6$ (by (*))

– $\varepsilon_{ijl}\varepsilon_{hkm}\delta_{ih}\delta_{jk} A_{lm} = \varepsilon_{ijl}\varepsilon_{ijm} A_{lm} = (\underbrace{\delta_{jj}}_{3}\delta_{lm} - \underbrace{\delta_{jm}\delta_{lj}}_{\delta_{lm}}) A_{lm} =$

$$= 2\delta_{lm} A_{lm} = 2 A_{ll} = 2\, Tr(\underline{\underline{A}})$$

– there are three terms in this form; so finally

$$\det(\underline{\underline{1}} + \phi \underline{\underline{A}}) = \frac{1}{6} \left[6 + \phi \cdot 3 \cdot 2\, Tr(\underline{\underline{A}}) \right] = 1 + \phi\, Tr(\underline{\underline{A}}) \quad q.e.d.$$

Note: This result can be used to show that $\det(\exp \underline{\underline{A}}) = \exp(Tr(\underline{\underline{A}}))$; indeed

$$\det(\exp \underline{\underline{A}}) = \det \left[\lim_{n \to +\infty} (\underline{\underline{1}} + \underline{\underline{A}}/n)^n \right] = \lim_{n \to +\infty} \det \left[(\underline{\underline{1}} + \underline{\underline{A}}/n)^n \right] = \lim_{n \to +\infty} \left[\det(\underline{\underline{1}} + \underline{\underline{A}}/n) \right]^n =$$

$Tr(\underline{\underline{A}})$: Trace of square, $n \times n$ matrix $\underline{\underline{A}}$ is $\sum_{i=1}^{n} A_{ii}$ (sum of diagonal elements)

$$= \lim_{n\to+\infty} \left[1 + \mathrm{Tr}(\underline{\underline{A}})/n \right]^n = \exp\left[\mathrm{Tr}(\underline{\underline{A}}) \right]$$

remember that $\lim_{n\to+\infty} (1+a/n)^n = \exp(a)$

[Warning: One should justify the correctness of taking and manipulating the limits.]